MATH 226: Differential Equations

Some Notes on Assignment 24

Find a power series solution for each of the following differential equations where

$$y = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + ...$$

$$\text{and } y' = a_1 + 2a_2x + 3a_3x^2 + ... + (n+1)a_{n+1}x^n + ...$$

1. $y' = 3 - 2y$ which we can write as $y' + 2y" = 3$
   Solution: The constant term on each side is 3; thus $a_1 + 2a_0 = 3$ or $a_0 = \frac{3-a_1}{2}$ For $n \geq 1$, the coefficient of x^n on the left is $(n+1)a_{n+1} + 2a_n$ and this difference must be 0. Thus

$$a_{n+1} = -2\frac{a_n}{n+1} \text{ is our recurrence relation for } n \geq 1$$

   Hence $a_2 = -2\frac{a_1}{2}, a_3 = -2\frac{a_2}{3} = \frac{-2}{3}(-2)\frac{a_1}{2} = \frac{(-2)^2a_1}{3!}, a_4 = -2\frac{a_3}{4} = \frac{(-2)^3a_1}{4!}, a_n = \frac{(-2)^{n-1}a_1}{n!} = \frac{(-2)^na_1}{2\times n!}$

   We can write the solution as the power series

$$y = a_0 + a_1x + \sum_{n=2}^{\infty}\frac{(-2)^na_1}{2\times n!}x^n = \frac{3-a_1}{2} + a_1x + a_1\sum_{n=2}^{\infty}\frac{(-2)^n}{2\times n!}x^n = \frac{3-a_1}{2} + a_1x + \frac{a_1}{2}\sum_{n=2}^{\infty}\frac{(-2x)^n}{n!}$$

   where $a_1 = 3 - 2a_0$. There are other equivalent ways to write this power series; for example, in terms of a_0. One form would begin

$$y = a+(3-2a)x+(-3+2a)x^2+\left(2-\frac{4}{3}a\right)x^3+\left(-1+\frac{2}{3}a\right)x^4+\left(\frac{2}{5}-\frac{4}{15}a\right)x^5+.. \text{ where } a = a_0 = y(0)$$

   but it's hard to see a general pattern here.
   Note that we can obtain an exact solution using the integrating factor e^{2x}. We obtain

$$y = \frac{3}{2} + \left(y(0) - \frac{3}{2}\right)e^{-2x}$$

2. $y' = x + y$
   Solution: The right hand side has the power series

$$x + y = a_0 + (a_1+1)x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{b+2} + ...$$

   Comparing coefficients of like terms, we have

$$a_1 = a_0, 2a_2 = a_1 + 1, 3a_3 = a_2, 4a_4 = a_3, ...(n+1)a_{n+1} = a_n, ...$$

$$\text{So } a_1 = a_0, a_2 = \frac{a_1+1}{2} = \frac{a_0+1}{2} \text{ and}$$

$$a_{n+1} = \frac{a_n}{n+1}, \text{ for } n \geq 3 \text{ is the recurrence relation.}$$

$$\text{Thus } a_3 = \frac{a_2}{3} = \frac{a_0+1}{3\times 2} = \frac{a_0+1}{3!},\ a_4 = \frac{a_3}{4} = \frac{a_0+1}{4\times 3!} = \frac{a_0+1}{4!},\ ...a_n = \frac{a_0+1}{n!}$$

   We have the power series (with $a_0 = a$)

$$y = \mathbf{a + ax + (a+1)\frac{x^2}{2!} + (a+1)\frac{x^3}{3!} + (a+1)\frac{x^4}{4!}} + ... + \mathbf{(a+1)\frac{x^n}{n!}} + ...$$
$$= (a+1-1) + (a+1-1)x + (a+1)\frac{x^2}{2!} + (a+1)\frac{x^3}{3!} + (a+1)\frac{x^4}{4!} + ... + (a+1)\frac{x^n}{n!} + ...$$
$$= -1 - x + (a+1)\left[1 + x + x^2/2! + x^3/3! + ... + x^n/n! + ...\right] = -1 - x + (a+1)e^x$$

   which exactly the same as the solution obtained by treating $y' = x + y$ as a first order linear differential equation and using an integrating factor.

3. $y' = xy$

Solution: Here

$$xy = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + ... + a_{n-1}x^n + ..$$

and

$$y' = a_1 + 2a_2x + 3a_3x^+ ... + (n+1)a_{n+1}x^n + ...$$

Equating coefficients of the x^n term:

$$(n+1)a_{n+1} = a_{n-1} \text{ so } a_{n+1} = \frac{a_{n-1}}{n+1}$$

All the terms with even index will be multiples of a_0

All the terms with odd index will be multiples of a_1 but $a_1 = 0$ since a_1 is the constant term in y' and 0 is the constant term in xy. Thus all odd terms are 0 and

$$a_2 = \frac{a_0}{2},\ a_4 = \frac{a_2}{4} = \frac{a_0}{4 \times 2},\ a_6 = \frac{a_4}{6} = \frac{a_0}{6 \times 4 \times 2} = \frac{a_0}{2^3 \times 3!}$$

$$\text{and, in general } a_{2n} = \frac{a_0}{2^n \times n!}$$

$$\text{So } \mathbf{y} = \mathbf{a_0}\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{x^2} + \frac{\mathbf{1}}{\mathbf{2^2} \times \mathbf{2!}}\mathbf{x^4} + \frac{\mathbf{1}}{\mathbf{2^3} \times \mathbf{3!}}\mathbf{x^6} + ...\right)$$

Note that we can obtain an exact solution by separation of variables

$$\frac{y'}{y} = x \text{ implies } \ln y = \frac{x^2}{2} + C \text{ so } y = a_0e^{x^2/2}$$

Now the power series for $e^{x^2/2}$ is

$$1 + \frac{x^2}{2} + \frac{(x^2/2)^2}{2!} + \frac{(x^2/2)^3}{3!} + \frac{(x^2/2)^4}{4!} + ...+$$

4. $y'' + y = 0$ so $y'' = -y$.

Solution: Note that $\{\sin x, \cos x\}$ is a linearly independent pair of solutions so every solution is a linear combination of these two. Our power series solution should be consistent with this observation.

Equating coefficients of like degree terms, we have

$$(n+2)(n+1)a_{n+2} = -a_n \text{ so } a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

The even degree terms will be multiples of a_0:

$$a_2 = -\frac{a_0}{2 \times 1},\ a_4 = -\frac{a_2}{4 \times 3} = \frac{a_0}{4!},\ a_6 = -\frac{a_0}{6!},\ a_8 = \frac{a_0}{8!}, ...a_{2n} = \frac{(-1)^n a_0}{(2n)!}, ...$$

The odd degree terms will be multiples of a_1:

$$a_3 = -\frac{a_1}{3 \times 2},\ a_5 = -\frac{a_3}{5 \times 4} = \frac{a_1}{5!},\ a_7 = -\frac{a_1}{7!},\ a_9 = \frac{a_1}{9!},\ a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, ...$$

Thus the solution has the form

$$\mathbf{y} = \mathbf{a_0}\sum_{\mathbf{n=0}}^{\infty}\frac{(-\mathbf{1})^{\mathbf{n}}}{(\mathbf{2n})!}\mathbf{x^{2n}} + \mathbf{a_1}\sum_{\mathbf{n=0}}^{\infty}\frac{(-\mathbf{1})^{\mathbf{n}}}{(\mathbf{2n+1})!}\mathbf{x^{2n+1}} = \mathbf{a_0}\cos\mathbf{x} + \mathbf{a_1}\sin\mathbf{x}$$

5. $xy'' + y' + xy = 0$

Solution:

Coefficient of x^n in xy is a_{n-1}

Coefficient of x^n in y' is $(n+1)a_{n+1}$

Coefficient of x^n in xy'' is $(n+1)(n)a_{n+1}$
Thus coefficient of x^n in $xy''+y'+xy$ is $(n+1)(n)a_{n+1}+(n+1)a_{n+1}+a_{n-1} = (n+1)^2a_{n+1}+a_{n-1}$

$$\text{So the recurrence relation is } a_{n+1} = \frac{-a_{n-1}}{(n+1)^2}$$

The constant term in the power series is a_1 and the constant term in 0 is 0 so a_1 and consequently all the odd indexed terms will be 0. For the even indexed terms (using $a = a_0$), we have

$$a_2 = -\frac{a}{2^2},\ a_4 = -\frac{a}{4^2} = \frac{a}{4^2 \times 2^2} = \frac{a}{(2^2 \times 2!)^2},\ a_6 = -\frac{a_4}{6^2} = -\frac{a}{(2^3 3!)^2}, ..., a_{2n} = \frac{(-1)^n a}{(2^n n!)^2}$$

and the power series solution is

$$\mathbf{y = y(0)\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2^n n!)^2}}$$