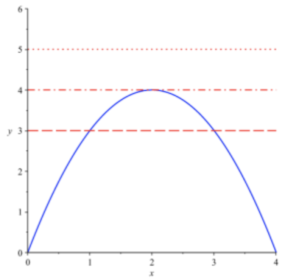
MATH 226: *Notes on Assignment 21*

7.3 Bifurcation Exercises

Feedback Problems: 12, 15, 17

12.(a) Below we sketch some nullclines, corresponding to $\alpha = 2, 8/3$ and $10/3$.



(b) The critical points are solutions of

$$\begin{aligned} 3\alpha/2 - y &= 0 \\ -4x + y + x^2 &= 0. \end{aligned}$$

The solutions of these equations are $\left(2 \pm \sqrt{4 - 3\alpha/2}, 3\alpha/2\right)$ and exist for $\alpha \leq 8/3$.

(c) For $\alpha = 2$, the critical points are $(1, 3)$ and $(3, 3)$. The Jacobian matrix is

$$\mathbf{J}(x, y) = \begin{pmatrix} 0 & -1 \\ -4 + 2x & 1 \end{pmatrix}.$$

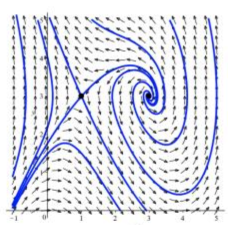
At $(1, 3)$,

$$\mathbf{J}(1, 3) = \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda = -1, 2$. Since they are of opposite sign $(1, 3)$ is a saddle, which is unstable. At $(3, 3)$,

$$\mathbf{J}(3, 3) = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}.$$

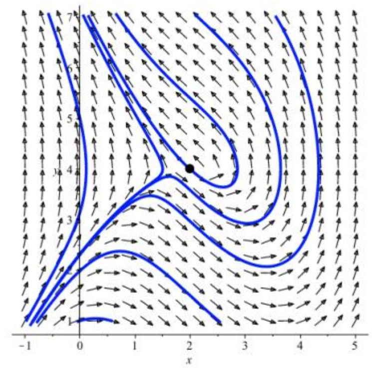
The eigenvalues are $\lambda = (1 \pm i\sqrt{7})/2$. Therefore, $(3, 3)$ is an unstable spiral.



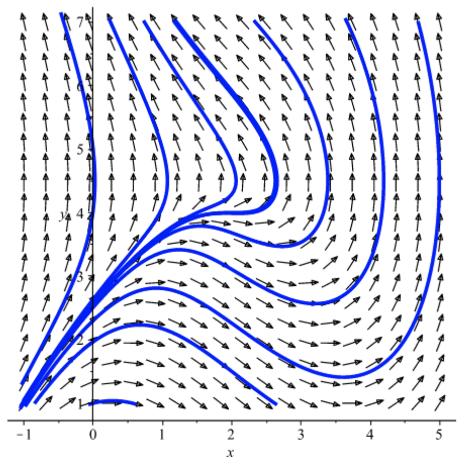
(d) The bifurcation value is $\alpha_0 = 8/3$. At this value α_0, the critical point is $(2, 4)$. The Jacobian matrix is

$$\mathbf{J}(2, 4) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda = 0, 1$.



(e) Below we show the phase portrait for $\alpha = 3$.



15.(a) The equation $x' = 0$ implies $x = -3$ or $x - y = 1$. The equation $y' = 0$ implies $y = 1$ or $x + \alpha y = -1$. Solving these equations, we see that the critical points are $(2,1)$, $(-3,1)$, $(-3, 2/\alpha)$, and $((-1+\alpha)/(1+\alpha), -2/(1+\alpha))$.

(b) When $\alpha_0 = 2$, the second and third critical points listed above coincide. When $\alpha_0 = -3$, the first and fourth critical points listed above coincide, but here we are only considering $\alpha > 0$. Therefore, $\alpha_0 = 2$.

(c) Here, we have $F(x,y) = (3+x)(1-x+y)$ and $G(x,y) = (y-1)(1+x+\alpha y)$. Therefore, the Jacobian matrix for this system is

$$\mathbf{J}(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -2-2x+y & 3+x \\ y-1 & 1+x+2\alpha y-\alpha \end{pmatrix}.$$

We will look at the linear systems near the second and third critical points above, namely $(-3,1)$ and $(-3,2/\alpha)$. Near the critical point $(-3,1)$, the Jacobian matrix is

$$\mathbf{J}(-3,1) = \begin{pmatrix} F_x(-3,1) & F_y(-3,1) \\ G_x(-3,1) & G_y(-3,1) \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -2+\alpha \end{pmatrix}$$

and the corresponding linear system near $(-3,1)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -2+\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x+3$ and $v = y-1$. Near the critical point $(-3,2/\alpha)$, the Jacobian matrix is

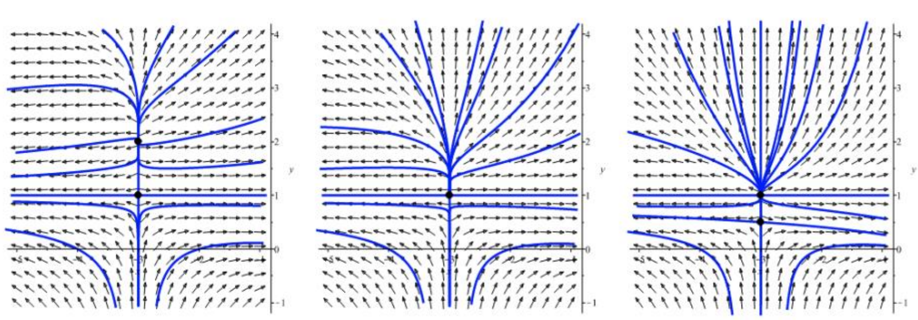
$$\mathbf{J}(-3,2/\alpha) = \begin{pmatrix} F_x(-3,2/\alpha) & F_y(-3,2/\alpha) \\ G_x(-3,2/\alpha) & G_y(-3,2/\alpha) \end{pmatrix} = \begin{pmatrix} 4+2/\alpha & 0 \\ -1+2/\alpha & 2-\alpha \end{pmatrix}$$

and the corresponding linear system near $(-3,2/\alpha)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4+2/\alpha & 0 \\ -1+2/\alpha & 2-\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x+3$ and $v = y-2/\alpha$. The eigenvalues for the linearized system near $(-3,1)$ are given by $\lambda = 5, -2+\alpha$. The eigenvalues for the linearized system near $(-3,2/\alpha)$ are given by $\lambda = 4+2/\alpha, 2-\alpha$. In part (b), we determined that the bifurcation point was $\alpha_0 = 2$. Here, we see that if $\alpha > 2$, then $(-3,1)$ will have two positive eigenvalues associated with it, and, therefore, be an unstable node, while $(-3,2/\alpha)$ will have eigenvalues of opposite signs, and, therefore, by an unstable saddle point. If $\alpha < 2$, then $(-3,1)$ will have eigenvalues of the opposite sign, and, therefore, be an unstable saddle point, while $(-3,2/\alpha)$ will have two positive eigenvalues, and, therefore, be an unstable node.

(d) The phase portraits below are for $\alpha = 1, 2$ and 4, respectively.



17.(a) The critical points need to satisfy the system of equations

$$\begin{aligned} x(4-x-y) &= 0 \\ y(2+2\alpha-y-\alpha x) &= 0. \end{aligned}$$

The four critical points are $(0,0)$, $(0,2+2\alpha)$, $(4,0)$, and $(2,2)$.

(b) The Jacobian matrix is given by

$$\mathbf{J}(x,y) = \begin{pmatrix} 4-2x-y & -x \\ -\alpha y & 2+2\alpha-2y-\alpha x \end{pmatrix}.$$

Therefore, at $(2,2)$,

$$\mathbf{J}(2,2) = \begin{pmatrix} -2 & -2 \\ -2\alpha & -2 \end{pmatrix}.$$

For $\alpha = 0.75$,

$$\mathbf{J}(2,2) = \begin{pmatrix} -2 & -2 \\ -3/2 & -2 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = -2 \pm \sqrt{3}$. Since both of these eigenvalues are negative, for $\alpha = 0.75$, $(2,2)$ is a stable node, which is asymptotically stable. For $\alpha = 1.25$,

$$\mathbf{J}(2,2) = \begin{pmatrix} -2 & -2 \\ -5/2 & -2 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = -2 \pm \sqrt{5}$. Since these eigenvalues are of opposite sign, for $\alpha = 1.25$, $(2,2)$ is a saddle point which is unstable. In general, the eigenvalues at $(2,2)$ are given by $-2 \pm 2\sqrt{\alpha}$. The nature of the critical point will change when $2\sqrt{\alpha} = 2$; that is, at $\alpha_0 = 1$. At this value of α, the number of negative eigenvalues changes from two to one.

(c) From the Jacobian matrix in part (b), we see that the approximate linear system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & -2 \\ -2\alpha & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

(d) As shown in part (b), the value $\alpha_0 = 1$.

(e) In the three phase portraits, we take $\alpha = 0.75, 1$ and 1.25, respectively.