MATH 226: *Notes on Assignment 16*

6.5 Fundamental Matrices and the Exponential of a Matrix

Practice Problems: 1, 6, 9, 13, 15, 16
Feedback Problems: 13, 16

1. We find the eigenvalues first.

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$$

implies $\det(A-\lambda I) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$. Thus the eigenvalues are $\lambda = 2, -1$. For $\lambda = 2$,

$$A - \lambda I = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix},$$

and then

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = 2$. Therefore,

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

is a solution of the system. For $\lambda = -1$,

$$A - \lambda I = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix},$$

and then

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is an eigenvector for $\lambda = -1$. Therefore,

$$\mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

is a solution of the system. We conclude that

$$\mathbf{X}(t) = \begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}$$

is a fundamental matrix for this system. The fundamental matrix e^{At} is given by $e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$. Thus

$$e^{At} = \frac{1}{3}\begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -e^{-t}+4e^{2t} & 2e^{-t}-2e^{2t} \\ -2e^{-t}+2e^{2t} & 4e^{-t}-e^{2t} \end{pmatrix}.$$

6. We find the eigenvalues first.

$$A - \lambda I = \begin{pmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix}$$

implies $\det(A - \lambda I) = \lambda^2 + 2\lambda + 5$. Thus the eigenvalues are $\lambda = -1 \pm 2i$. For $\lambda = -1 + 2i$,

$$A - \lambda I = \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \to \begin{pmatrix} 1 & -2i \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = -1 + 2i$. Then

$$\mathbf{x}_1(t) = e^{(-1+2i)t} \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

is a solution of the system. Taking the real and imaginary parts of $\mathbf{u}(t)$, we have the two linearly independent real-valued solutions

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}$$

and

$$\mathbf{x}_2(t) = e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}.$$

We conclude that

$$\mathbf{X}(t) = e^{-t} \begin{pmatrix} -2\sin 2t & 2\cos 2t \\ \cos 2t & \sin 2t \end{pmatrix}$$

is a fundamental matrix for this system. The fundamental matrix e^{At} is given by $e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$. Therefore,

$$e^{At} = \frac{1}{2}e^{-t} \begin{pmatrix} -2\sin 2t & 2\cos 2t \\ \cos 2t & \sin 2t \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}e^{-t} \begin{pmatrix} 2\cos 2t & -4\sin 2t \\ \sin 2t & 2\cos 2t \end{pmatrix}.$$

9. We find the eigenvalues first.

$$A - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix}$$

implies $\det(A - \lambda I) = \lambda^2 - 1$. Therefore, the eigenvalues are $\lambda = 1, -1$. For $\lambda = 1$,

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = 1$. Thus

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

is a solution of the system. For $\lambda = -1$,

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \to \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

is an eigenvector for $\lambda = -1$. Thus

$$\mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

is a solution of the system. We conclude that

$$\mathbf{X}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

is a fundamental matrix for this system. The fundamental matrix e^{At} is given by $e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$. Thus

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{pmatrix}.$$

13. The eigenvalues of this matrix are $\lambda = -1, -2, 2$. For $\lambda = -1$,

$$A - \lambda I = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$$

is an eigenvector for $\lambda = -1$. Thus

$$\mathbf{x}_1(t) = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} e^{-t}$$

is a solution of the system. For $\lambda = -2$,

$$A - \lambda I = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix} \to \begin{pmatrix} 7 & 0 & 4 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$\mathbf{v}_2 = \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

is an eigenvector for $\lambda = -2$. Thus

$$\mathbf{x}_2(t) = \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} e^{-2t}$$

is a solution of the system. For $\lambda = 2$,

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

is an eigenvector for $\lambda = 2$. Thus

$$\mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$$

is a solution of the system. We conclude that

$$\mathbf{X}(t) = \begin{pmatrix} 3e^{-t} & 4e^{-2t} & 0 \\ -4e^{-t} & -5e^{-2t} & e^{2t} \\ -2e^{-t} & -7e^{-2t} & -e^{2t} \end{pmatrix}$$

is a fundamental matrix for this system. The fundamental matrix e^{At} is given by $e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$. Therefore,

$$\begin{aligned} e^{At} &= \begin{pmatrix} 3e^{-t} & 4e^{-2t} & 0 \\ -4e^{-t} & -5e^{-2t} & e^{2t} \\ -2e^{-t} & -7e^{-2t} & -e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 1/3 & 1/3 \\ -1/2 & -1/4 & -1/4 \\ 3/2 & 13/12 & 1/12 \end{pmatrix} \\ &= \begin{pmatrix} -2e^{-2t} + 3e^{-t} & -e^{-2t} + e^{-t} & -e^{-2t} + e^{-t} \\ 5e^{-2t}/2 - 4e^{-t} + 3e^{2t}/2 & 5e^{-2t}/4 - 4e^{-t}/3 + 13e^{2t}/12 & 5e^{-2t}/4 - 4e^{-t}/3 + e^{2t}/12 \\ 7e^{-2t}/2 - 2e^{-t} - 3e^{2t}/2 & 7e^{-2t}/4 - 2e^{-t}/3 - 13e^{2t}/12 & 7e^{-2t}/4 - 2e^{-t}/3 - e^{2t}/12 \end{pmatrix}. \end{aligned}$$

15. We find that for this A matrix

$$e^{At} = \frac{1}{3}e^{-3t}\begin{pmatrix} 3\cos 3t & -9\sin 3t \\ \sin 3t & 3\cos 3t \end{pmatrix}.$$

Therefore, the solution of the initial value problem is given by

$$\mathbf{x} = e^{At}\mathbf{x}_0 = \frac{1}{3}e^{-3t}\begin{pmatrix} 3\cos 3t & -9\sin 3t \\ \sin 3t & 3\cos 3t \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = e^{-3t}\begin{pmatrix} 4\cos 3t - 3\sin 3t \\ \cos 3t + (4/3)\sin 3t \end{pmatrix}.$$

16. The fundamental matrix found in problem 9 was

$$e^{At} = \frac{1}{2}\begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{pmatrix}.$$

Therefore, the solution of the initial value problem is given by

$$\mathbf{x} = e^{At}\mathbf{x}_0 = \frac{1}{2}\begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 7e^t - 3e^{-t} \\ 7e^t - 9e^{-t} \end{pmatrix}.$$

6.6 Nonhomogeneous Linear Systems

Practice Problems: 2, 5, 9
Feedback Problems: 5

2. The eigenvalues of

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$$

are given by $\lambda_1 = 1$ and $\lambda_2 = -1$. Corresponding eigenvectors are given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Therefore, two linearly independent solutions are given by

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t, \qquad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}.$$

and

$$\mathbf{X}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

is a fundamental matrix. In order to calculate the general solution, we need to calculate $\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)ds$. We see that

$$\mathbf{X}^{-1}(s) = \frac{1}{2}\begin{pmatrix} 3e^{-s} & -e^{-s} \\ -e^{s} & e^{s} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)\,ds &= \frac{1}{2}\int_{t_1}^{t} \begin{pmatrix} 3e^{-s} & -e^{-s} \\ -e^{s} & e^{s} \end{pmatrix}\begin{pmatrix} e^{s} \\ s \end{pmatrix} ds = \frac{1}{2}\int_{t_1}^{t} \begin{pmatrix} 3 - se^{-s} \\ -e^{2s} + se^{s} \end{pmatrix} ds \\
&= \frac{1}{2}\begin{pmatrix} 3t + te^{-t} + e^{-t} \\ -e^{2t}/2 + te^{t} - e^{t} \end{pmatrix} + \mathbf{c}.
\end{aligned}$$

Then the general solution will be given by

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)\,ds \\
&= \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\mathbf{c} + \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} 3t + te^{-t} + e^{-t} \\ -e^{2t}/2 + te^{t} - e^{t} \end{pmatrix} + \mathbf{c}\right] \\
&= c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} (3t/2 - 1/4)e^{t} + t \\ (3t/2 - 3/4)e^{t} + 2t - 1 \end{pmatrix}.
\end{aligned}$$

5. The eigenvalues of

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

are given by $\lambda_1 = -3$ and $\lambda_2 = 2$. Corresponding eigenvectors are given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, two linearly independent solutions of the homogeneous equation are given by

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}, \qquad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

and

$$\mathbf{X}(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

is a fundamental matrix for this equation. In order to calculate the general solution, we need to calculate $\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)\,ds$. We see that

$$\mathbf{X}^{-1}(s) = \frac{1}{5}\begin{pmatrix} e^{3s} & -e^{3s} \\ 4e^{-2s} & e^{-2s} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)\,ds &= \frac{1}{5}\int_{t_1}^{t} \begin{pmatrix} e^{3s} & -e^{3s} \\ 4e^{-2s} & e^{-2s} \end{pmatrix} \begin{pmatrix} e^{-2s} \\ -2e^{s} \end{pmatrix} ds \\
&= \frac{1}{5}\int_{t_1}^{t} \begin{pmatrix} e^{s} + 2e^{4s} \\ 4e^{-4s} - 2e^{-s} \end{pmatrix} ds = \frac{1}{5}\begin{pmatrix} e^{t} + e^{4t}/2 \\ -e^{-4t} + 2e^{-t} \end{pmatrix} + \mathbf{c}.
\end{aligned}$$

Then the general solution will be given by

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)\,ds \\
&= \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}\mathbf{c} + \frac{1}{5}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}\begin{pmatrix} e^{t} + e^{4t}/2 \\ -e^{-4t} + 2e^{-t} \end{pmatrix} \\
&= c_1 e^{-3t}\begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} e^{t}/2 \\ -e^{-2t} \end{pmatrix}.
\end{aligned}$$

9. The eigenvalues of A are given by $\lambda_1 = -2$, $\lambda_2 = -1$ and $\lambda_3 = 1$ with eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore, three solutions of the homogeneous equation are given by

$$\mathbf{x}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3(t) = e^{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and

$$\mathbf{X}(t) = \begin{pmatrix} e^{-2t} & -e^{-t} & e^{t} \\ -2e^{-2t} & 0 & e^{t} \\ e^{-2t} & e^{-t} & e^{t} \end{pmatrix}$$

is a fundamental matrix for this equation. Further,

$$\mathbf{X}^{-1}(s) = \frac{1}{6} \begin{pmatrix} e^{2s} & -2e^{2s} & e^{2s} \\ -3e^{s} & 0 & 3e^{s} \\ 2e^{-s} & 2e^{-s} & 2e^{-s} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)\, ds &= \frac{1}{6} \int_{t_1}^{t} \begin{pmatrix} e^{2s} & -2e^{2s} & e^{2s} \\ -3e^{s} & 0 & 3e^{s} \\ 2e^{-s} & 2e^{-s} & 2e^{-s} \end{pmatrix} \begin{pmatrix} 0 \\ -\sin s \\ 0 \end{pmatrix} ds \\
&= \frac{1}{6} \int_{t_1}^{t} \begin{pmatrix} 2e^{2s}\sin s \\ 0 \\ -2e^{-s}\sin s \end{pmatrix} ds = \frac{1}{6} \begin{pmatrix} -(2/5)e^{2t}\cos t + (4/5)e^{2t}\sin t \\ 0 \\ e^{-t}\cos t + e^{-t}\sin t \end{pmatrix} + \mathbf{c}.
\end{aligned}$$

Therefore, the general solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t_1}^{t} \mathbf{X}^{-1}(s)\mathbf{g}(s)\, ds = \begin{pmatrix} e^{-2t} & -e^{-t} & e^{t} \\ -2e^{-2t} & 0 & e^{t} \\ e^{-2t} & e^{-t} & e^{t} \end{pmatrix} \mathbf{c} \\
&+ \frac{1}{6} \begin{pmatrix} e^{-2t} & -e^{-t} & e^{t} \\ -2e^{-2t} & 0 & e^{t} \\ e^{-2t} & e^{-t} & e^{t} \end{pmatrix} \begin{pmatrix} -(2/5)e^{2t}\cos t + (4/5)e^{2t}\sin t \\ 0 \\ e^{-t}\cos t + e^{-t}\sin t \end{pmatrix} \\
&= c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} (1/10)\cos t + (3/10)\sin t \\ -(1/10)\sin t + (3/10)\cos t \\ (1/10)\cos t + (3/10)\sin t \end{pmatrix}.
\end{aligned}$$