MATH 226: *Notes on Assignment 15*

6.3 Homogeneous Linear Systems with Constant Coefficients

Practice Problems: 1, 2, 11*, 15*, 18*

1. We will write the system of equations in matrix form as $\mathbf{x}' = A\mathbf{x}$. Here, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -4 & 1 & 0 \\ 1 & -5 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

To solve this system, we need to compute the eigenvalues and eigenvectors of A. We have

$$A - \lambda I = \begin{pmatrix} -4-\lambda & 1 & 0 \\ 1 & -5-\lambda & 1 \\ 0 & 1 & -4-\lambda \end{pmatrix}.$$

Therefore, $\det(A - \lambda I) = -\lambda^3 - 13\lambda^2 - 54\lambda - 72 = -(\lambda+3)(\lambda+6)(\lambda+4)$. Thus the eigenvalues are $\lambda = -3, -4, -6$. First, $\lambda = -3$ implies

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = -3$, and, consequently,

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is a solution of the system. Next, $\lambda = -4$ implies

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

is an eigenvector for $\lambda = -4$, and, consequently,

$$\mathbf{x}_2(t) = e^{-4t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

is a solution of the system. Finally, $\lambda = -6$ implies

$$A - \lambda I = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = -6$, and, consequently,

$$\mathbf{x}_3(t) = e^{-6t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

is a solution of the system. Thus the general solution is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-6t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

2. We will write the system of equations in matrix form as $\mathbf{x}' = A\mathbf{x}$. Here, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 1 & 4 & 4 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

To solve this system, we need to compute the eigenvalues and eigenvectors of A. We have

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 4 & 4 \\ 0 & 3-\lambda & 2 \\ 0 & 2 & 3-\lambda \end{pmatrix}.$$

Therefore, $\det(A - \lambda I) = (1-\lambda)(\lambda^2 - 6\lambda + 5) = (1-\lambda)(\lambda - 5)(\lambda - 1)$. Thus the eigenvalues are $\lambda = 1, 5$. First, $\lambda = 1$ implies

$$A - \lambda I = \begin{pmatrix} 0 & 4 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

are linearly independent eigenvectors for $\lambda = 1$, and, consequently,

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_2(t) = e^t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

are solutions of the system. Next, $\lambda = 5$ implies

$$A - \lambda I = \begin{pmatrix} -4 & 4 & 4 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = -5$, and, consequently,

$$\mathbf{x}_3(t) = e^{5t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

is a solution of the system. Thus the general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

11. We need to find the eigenvalues and eigenvectors.

$$A - \lambda I = \begin{pmatrix} -1-\lambda & 0 & 3 \\ 0 & -2-\lambda & 0 \\ 3 & 0 & -1-\lambda \end{pmatrix}$$

implies $\det(A - \lambda I) = -(\lambda+2)(\lambda^2+2\lambda-8) = -(\lambda+2)(\lambda+4)(\lambda-2)$. Thus the eigenvalues are $\lambda = -2, -4, 2$. First, for $\lambda = -2$,

$$A - \lambda I = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is an eigenvector for $\lambda = -2$ and

$$\mathbf{x}_1(t) = e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is a solution of the system. Second, for $\lambda = -4$,

$$A - \lambda I = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

is an eigenvector for $\lambda = -4$ and

$$\mathbf{x}_2(t) = e^{-4t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

is a solution of the system. Last, for $\lambda = 2$,

$$A - \lambda I = \begin{pmatrix} -3 & 0 & 3 \\ 0 & -4 & 0 \\ 3 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = 2$ and

$$\mathbf{x}_3(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is a solution of the system. Thus the general solution is

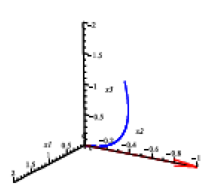
$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The initial condition implies

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.$$

The solution of this equation is $c_1 = -1$, $c_2 = 2$ and $c_3 = 0$. Therefore, the solution is

$$\mathbf{x}(t) = e^{-2t} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + 2e^{-4t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$



The solution tends to the origin approaching the eigenvector $(0, -1, 0)^T$ as $t \to \infty$.

15. We can see that

$$A - \lambda I = \begin{pmatrix} 2-\lambda & -4 & -3 \\ 3 & -5-\lambda & -3 \\ -2 & 2 & 1-\lambda \end{pmatrix}.$$

Therefore, $\det(A-\lambda I) = -\lambda^3 - 2\lambda^2 + \lambda + 2 = -(\lambda+2)(\lambda+1)(\lambda-1)$. Thus the eigenvalues are given by $\lambda = -2, -1, 1$. The corresponding eigenvectors are given as follows. For $\lambda_1 = -2$,

$$A - \lambda I = \begin{pmatrix} 4 & -4 & -3 \\ 3 & -3 & -3 \\ -2 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = -1$,

$$A - \lambda I = \begin{pmatrix} 3 & -4 & -3 \\ 3 & -4 & -3 \\ -2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_3 = 1$,

$$A - \lambda I = \begin{pmatrix} 1 & -4 & -3 \\ 3 & -6 & -3 \\ -2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Thus the general solution is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

If we want the solution to tend to $(0,0,0)^T$ as $t \to \infty$, we need $c_3 = 0$. That is, we need the initial condition $\mathbf{x}_0$ to satisfy

$$\mathbf{x}_0 = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

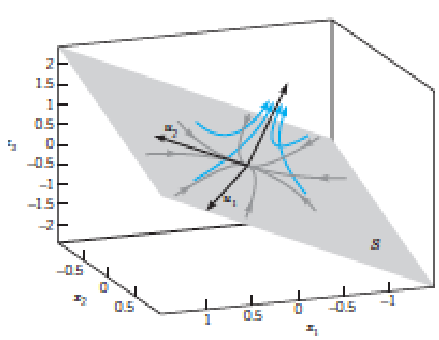
Therefore, letting

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

and letting

$$S = \{\mathbf{u} : \mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2, \ -\infty < a_1, a_2 < \infty\},$$

then for any $\mathbf{x}_0 \in S$, the solution $\mathbf{x}(t) \to (0,0,0)^T$ as $t \to \infty$.



If $\mathbf{x_0}$ is not in S, then $\mathbf{x}(t)$ approaches the lines determined by $\mathbf{v_3} = (1,1,-1)^T$ as $t \to \infty$

18. The eigenvalues are given by $\lambda = -1, -2, -3, -4$. Their associated eigenvectors are given by

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a fundamental set of solutions is given by

$$\{e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, e^{-3t}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, e^{-4t}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}.$$

6.4 Nondefective Matrices with Complex Eigenvalues

Practice Problems: 3, 5, 11, 15*
Feedback Problems: 3, 11

3. We find the eigenvalues first.

$$A - \lambda I = \begin{pmatrix} -\lambda & -2 & -1 \\ 1 & -1-\lambda & 1 \\ 1 & -2 & -2-\lambda \end{pmatrix}$$

implies $\det(A-\lambda I) = -\lambda^3 - 3\lambda^2 - 7\lambda - 5 = -(\lambda+1)(\lambda^2+2\lambda+5)$. Therefore, the eigenvalues are given by $\lambda = -1$ and $\lambda = -1 \pm 2i$. First, for $\lambda = -1$, we have

$$A - \lambda I = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 0 & 1 \\ 1 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

is an associated eigenvector, and

$$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

is a solution of the system. Next, for $\lambda = -1 + 2i$,

$$A - \lambda I = \begin{pmatrix} 1-2i & -2 & -1 \\ 1 & -2i & 1 \\ 1 & -2 & -1-2i \end{pmatrix} \to \begin{pmatrix} 1 & -2i & 1 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}$$

is an associated eigenvector. Further,

$$\mathbf{u}(t) = e^{(-1+2i)t} \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right]$$

is a solution of the system. We know that if $\mathbf{u}(t)$ is a solution, then $\mathrm{Re}(\mathbf{u})$ and $\mathrm{Im}(\mathbf{u})$ are also solutions. Consequently, we get the following two linearly independent solutions.

$$\mathbf{x}_2(t) = \mathrm{Re}(\mathbf{u}) = e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \\ \cos 2t \end{pmatrix}$$

and

$$\mathbf{x}_3(t) = \mathrm{Im}(\mathbf{u}) = e^{-t} \begin{pmatrix} \sin 2t \\ -\cos 2t \\ \sin 2t \end{pmatrix}.$$

We conclude that the general solution is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \\ \cos 2t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sin 2t \\ -\cos 2t \\ \sin 2t \end{pmatrix}.$$

5. We find the eigenvalues first.

$$A - \lambda I = \begin{pmatrix} -7-\lambda & 6 & -6 \\ -9 & 5-\lambda & -9 \\ 0 & -1 & -1-\lambda \end{pmatrix}$$

implies $\det(A - \lambda I) = -\lambda^3 - 3\lambda^2 - 12\lambda - 10 = -(1+\lambda)(\lambda^2 + 2\lambda + 10)$. Therefore, the eigenvalues are given by $\lambda = -1$ and $\lambda = -1 \pm 3i$. First, for $\lambda = -1$, we have

$$A - \lambda I = \begin{pmatrix} -6 & 6 & -6 \\ -9 & 6 & -9 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

is an associated eigenvector, and

$$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

is a solution of the system. Next, for $\lambda = -1 + 3i$,

$$A - \lambda I = \begin{pmatrix} -6-3i & 6 & -6 \\ -9 & 6-3i & -9 \\ 0 & -1 & -3i \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2+2i \\ 0 & 1 & 3i \\ 0 & 0 & 0 \end{pmatrix}$$

after elementary row operations. Therefore,

$$\mathbf{v}_2 = \begin{pmatrix} 2+2i \\ 3i \\ -1 \end{pmatrix}$$

is an associated eigenvector. Further,

$$\mathbf{u}(t) = e^{(-1+3i)t} \left[\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right]$$

is a solution of the system. We know that if $\mathbf{u}(t)$ is a solution, then $\mathrm{Re}(\mathbf{u})$ and $\mathrm{Im}(\mathbf{u})$ are also solutions. Consequently, we get the following two linearly independent solutions.

$$\mathbf{x}_2(t) = \mathrm{Re}(\mathbf{u}) = e^{-t} \begin{pmatrix} 2\cos 3t - 2\sin 3t \\ -3\sin 3t \\ -\cos 3t \end{pmatrix}$$

and

$$\mathbf{x}_3(t) = \mathrm{Im}(\mathbf{u}) = e^{-t} \begin{pmatrix} 2\sin 3t + 2\cos 3t \\ 3\cos 3t \\ -\sin 3t \end{pmatrix}.$$

We conclude that the general solution is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\cos 3t - 2\sin 3t \\ -3\sin 3t \\ -\cos 3t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 2\sin 3t + 2\cos 3t \\ 3\cos 3t \\ -\sin 3t \end{pmatrix}.$$

11.(a) Suppose that $c_1\mathbf{a} + c_2\mathbf{b} = 0$. Since $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of the vector $\mathbf{v}_1$, $\mathbf{a} = (\mathbf{v}_1 + \overline{\mathbf{v}}_1)/2$ and $\mathbf{b} = (\mathbf{v}_1 - \overline{\mathbf{v}}_1)/2i$. Therefore,

$$c_1(\mathbf{v}_1 + \overline{\mathbf{v}}_1) - ic_2(\mathbf{v}_1 - \overline{\mathbf{v}}_1) = 0,$$

which leads to

$$(c_1 - ic_2)\mathbf{v}_1 + (c_1 + ic_2)\overline{\mathbf{v}}_1 = 0.$$

(b) Since $\mathbf{v}_1$ and $\overline{\mathbf{v}}_1$ are linearly independent, we must have

$$\begin{aligned} c_1 - ic_2 &= 0 \\ c_1 + ic_2 &= 0. \end{aligned}$$

It follows that $c_1 = c_2 = 0$.

(c) Consider the equation $c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) = 0$. Using equation (4), we can then write

$$c_1 e^{\mu t_0}(\mathbf{a}\cos\nu t_0 - \mathbf{b}\sin\nu t_0) + c_2 e^{\mu t_0}(\mathbf{a}\sin\nu t_0 + \mathbf{b}\cos\nu t_0) = 0.$$

Rearranging the terms and dividing by the exponential,

$$(c_1 + c_2)\cos(\nu t_0)\mathbf{a} + (c_2 - c_1)\sin(\nu t_0)\mathbf{b} = 0.$$

From part (b), since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, it follows that

$$(c_1 + c_2)\cos(\nu t_0) = (c_2 - c_1)\sin(\nu t_0) = 0.$$

Without loss of generality, we may assume that the trigonometric factors are nonzero. We then conclude that $c_1 + c_2 = 0$ and $c_2 - c_1 = 0$, which leads to $c_1 = c_2 = 0$. Therefore, $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent at the point t_0 and therefore at every point.

15. The eigenvalues are given by $-1, 1, -1 \pm 4i$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ i \\ 1 \\ i \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ -i \\ 1 \\ -i \end{pmatrix}.$$

Therefore, a fundamental set of solutions is given by

$$\{e^{-t}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, e^{t}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, e^{(-1+4i)t}\begin{pmatrix} 1 \\ i \\ 1 \\ i \end{pmatrix}, e^{(-1-4i)t}\begin{pmatrix} 1 \\ -i \\ 1 \\ -i \end{pmatrix}\}.$$

In order to write the solutions as real-valued solutions, we look at the solution given from the eigenvalue $-1+4i$, and take the real and imaginary parts of that solution. In particular, for

$$\mathbf{u}(t) = e^{(-1+4i)t}\begin{pmatrix} 1 \\ i \\ 1 \\ i \end{pmatrix},$$

we have the two linearly independent real-valued solutions

$$\mathbf{x}_1(t) = \mathrm{Re}(\mathbf{u}(t)) = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\sin 4t\right]$$

and

$$\mathbf{x}_2(t) = \mathrm{Im}(\mathbf{u}(t)) = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sin 4t + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cos 4t\right].$$

Therefore, we conclude that a fundamental set of real-valued solutions is given by

$$\{e^{-t}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, e^{t}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, e^{-t}\begin{pmatrix} \cos 4t \\ -\sin 4t \\ \cos 4t \\ -\sin 4t \end{pmatrix}, e^{-t}\begin{pmatrix} \sin 4t \\ \cos 4t \\ \sin 4t \\ \cos 4t \end{pmatrix}\}.$$