MATH 226: *Notes on Assignment 13*

6.1 Definitions and Examples

Practice Problems: 2, 3, 4,5

2. First, we note that

$$\mathbf{x}' = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}.$$

At the same time, we calculate

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x} &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}. \end{aligned}$$

3. First, we see that

$$\Psi' = \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}.$$

At the same time,

$$\begin{aligned} \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \Psi &= \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}. \end{aligned}$$

4. Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$ and $x_4 = y'''$. Then

$$\begin{aligned}
x_1' &= y' = x_2\\
x_2' &= y'' = x_3\\
x_3' &= y''' = x_4\\
x_4' &= y'''' = -6y''' - 3y + t = -6x_4 - 3x_1 + t.
\end{aligned}$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix}.$$

5. Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. Then

$$\begin{aligned}
x_1' &= y' = x_2\\
x_2' &= y'' = x_3\\
x_3' &= y''' = -\frac{\sin t}{t} y'' - \frac{8}{t} y + \cos t = -\frac{\sin t}{t} x_3 - \frac{8}{t} x_1 + \cos t.
\end{aligned}$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8/t & 0 & -\sin t/t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix}.$$

6.2 Basic Theory of First Order Linear Systems

Practice Problems: 1, 3, 5, 7, 9, 11, 15

1. By Corollary 6.2.8, since the functions $p_1(t) = 5$, $p_4(t) = 4$ and $g(t) = t$ are continuous on $(-\infty, \infty)$, the solution is sure to exist for all $t \in (-\infty, \infty)$.

3. We will rewrite the equation and apply Corollary 6.2.8. We rewrite the equation as

$$y^{(4)} + \frac{e^t}{t(t-1)}y'' + \frac{7t^2}{t(t-1)}y = 0.$$

Since the coefficient functions $p_2(t) = \frac{e^t}{t(t-1)}$ and $p_4(t) = \frac{7t^2}{t(t-1)}$ are continuous for all $t \neq 0, 1$, the solution is sure to exist on the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

5. First, we rewrite the equation as

$$y^{(4)} + \frac{x+5}{x-1}y'' + \frac{\tan x}{x-1}y = 0.$$

The coefficient functions are $p_2(x) = \frac{x+5}{x-1}$ and $p_4(x) = \frac{\tan x}{x-1}$. These functions are continuous for all $x \neq 1, \pm\pi/2, \pm 3\pi/2, \ldots$. Therefore, the solution is sure to exist on the intervals $\ldots, (-3\pi/2, -\pi/2), (-\pi/2, 1), (1, \pi/2), (\pi/2, 3\pi/2), \ldots$

7. First, we let

$$\mathbf{X}(t) = \begin{pmatrix} e^t & e^{-t} & 2e^{4t} \\ 2e^t & -2e^{-t} & 2e^{4t} \\ -e^t & e^{-t} & -8e^{4t} \end{pmatrix}.$$

For the first method of computing $W[\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]$, we begin by computing $|\mathbf{X}(t)|$. We see that

$$|\mathbf{X}(t)| = e^t(16e^{3t} - 2e^{3t}) - e^{-t}(-16e^{5t} + 2e^{5t}) + 2e^{4t}(2-2) = 14e^{4t} + 14e^{4t} = 28e^{4t}.$$

Then, evaluating $|\mathbf{X}(0)|$, we have $|\mathbf{X}(0)| = 28$.
For the second method, we start by evaluating $\mathbf{X}(0)$. We see that

$$\mathbf{X}(0) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -2 & 2 \\ -1 & 1 & -8 \end{pmatrix}.$$

Then, $|\mathbf{X}(0)| = 28$.

9. First, $\mathbf{x}_i' = A\mathbf{x}_i$ for $i = 1, 2, 3$, thus the $\mathbf{x}_i$'s are solutions. Next, we calculate $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](0)$. Now

$$\mathbf{X}(t) = \begin{pmatrix} e^{-t} & e^{-t} & 2e^{8t} \\ 0 & -4e^{-t} & e^{8t} \\ -e^{-t} & e^{-t} & 2e^{8t} \end{pmatrix}$$

implies

$$\mathbf{X}(0) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -4 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Therefore,

$$W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](0) = \det \mathbf{X}(0) = 1(-8-1) - 1(1+8) = -18.$$

Since $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](0) \neq 0$, these functions are linearly independent and form a fundamental set of solutions.

11. We compute: $y_1 = 1$ gives $y_1''' + y_1' = 0$, $y_2 = \cos t$ gives $y_2''' + y_2' = \sin t - \sin t = 0$, and $y_3 = \sin t$ gives $y_3''' + y_3' = -\cos t + \cos t = 0$. Therefore, y_1, y_2, y_3 are all solutions of the differential equation. We now compute their Wronskian. We have

$$W[y_1, y_2, y_3] = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix},$$

which implies $W[y_1, y_2, y_3] = 1(\sin^2 t + \cos^2 t) = 1$.

15. We compute: $y_1 = 1$ gives $xy_1''' - y_1'' = 0$, $y_2 = x$ gives $xy_2''' - y_2'' = 0$, and $y_3 = x^3$ gives $xy_3''' - y_3'' = 6x - 6x = 0$. Therefore, y_1, y_2, y_3 are all solutions of the differential equation. We now compute their Wronskian. We have

$$W[y_1, y_2, y_3] = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix},$$

which implies $W[y_1, y_2, y_3] = 6x$.