MATH 226: *Notes on Assignment 12*

3.5 Repeated Eigenvalues

Practice Problems: 1*, 3*, 5*, 7*,9*

1.

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix}$$

implies $\det(A - \lambda I) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. Therefore, $\lambda = 1$ is the only eigenvalue. Now $\lambda = 1$ implies

$$A - \lambda I = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = 1$ and

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is one solution. Next, we need to look for a solution $\mathbf{w}$ of

$$(A - I)\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Plugging in for $A - I$, this equation becomes

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We see that

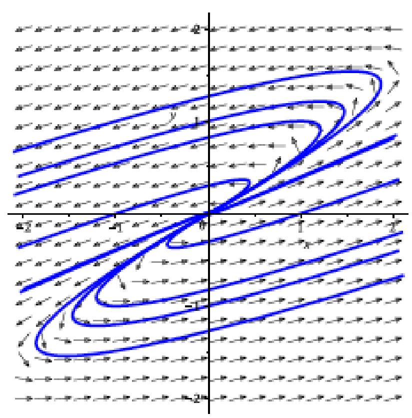
$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a solution of this equation. Therefore,

$$\mathbf{x}_2(t) = te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is also a solution of our original system. Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$



Nonzero solutions grow as $t \to \infty$.

3.

$$A - \lambda I = \begin{pmatrix} -3/2 - \lambda & 1 \\ -1/4 & -1/2 - \lambda \end{pmatrix}$$

implies $\det(A - \lambda I) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. Therefore, $\lambda = -1$ is the only eigenvalue. Now $\lambda = -1$ implies

$$A - \lambda I = \begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = -1$ and

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is one solution. Next, we need to look for a solution $\mathbf{w}$ of

$$(A + I)\,\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Plugging in for $A + I$, this equation becomes

$$\begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We see that

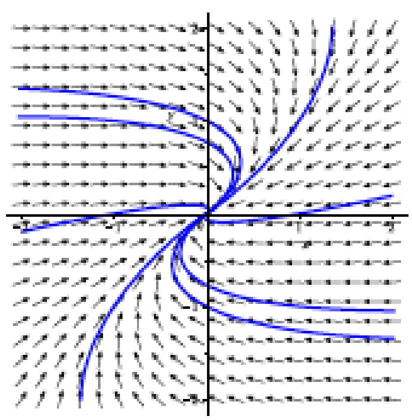
$$\mathbf{w} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

is a solution of this equation. Therefore,

$$\mathbf{x}_2(t) = te^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

is also a solution of our original system. Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[te^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right].$$



The solutions approach the origin as $t \to \infty$.

5.

$$A - \lambda I = \begin{pmatrix} -1-\lambda & -1/2 \\ 2 & -3-\lambda \end{pmatrix}$$

implies $\det(A - \lambda I) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$. Therefore, $\lambda = -2$ is the only eigenvalue. Now $\lambda = -2$ implies

$$A - \lambda I = \begin{pmatrix} 1 & -1/2 \\ 2 & -1 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is an eigenvector for $\lambda = -2$ and

$$\mathbf{x}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is one solution. Next, we need to look for a solution $\mathbf{w}$ of

$$(A + 2I)\,\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Plugging in for $A + 2I$, this equation becomes

$$\begin{pmatrix} 1 & -1/2 \\ 2 & -1 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We see that

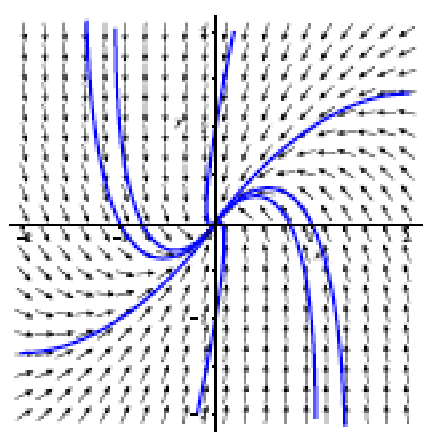
$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a solution of this equation. Therefore,

$$\mathbf{x}_2(t) = te^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is also a solution of our original system. Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[te^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$



7. The characteristic equation is $(\lambda+3)^2 = 0$. Therefore, the only eigenvalue is $\lambda = -3$. A corresponding eigenvector is given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is one solution of the system. Next, we need to look for a solution $\mathbf{w}$ of

$$(A+3I)\,\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Plugging in for $A+3I$, this equation becomes

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We see that

$$\mathbf{w} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

is a solution of this equation. Therefore,

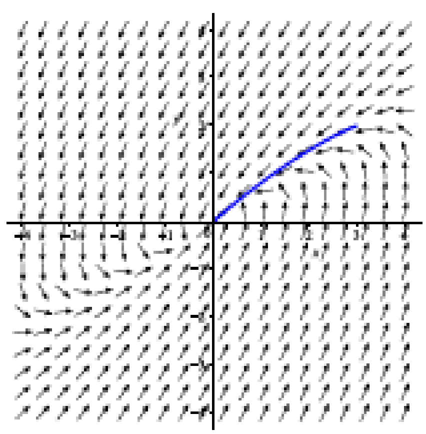
$$\mathbf{x}_2(t) = te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

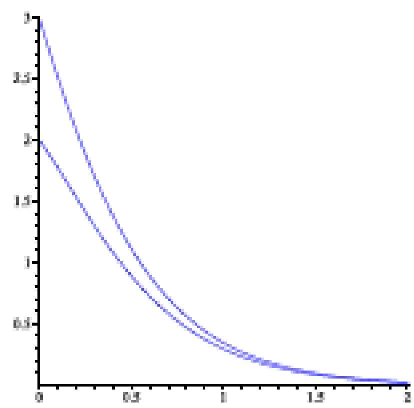
is also a solution of our original system. Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left[te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}\right].$$

The initial condition implies that $c_1 = 2$ and $c_2 = 4$. Therefore, the solution of the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} 3+4t \\ 2+4t \end{pmatrix} e^{-3t}.$$





9. The characteristic equation is $(\lambda - 1/2)^2 = 0$. Therefore, the only eigenvalue is $\lambda = 1/2$. A corresponding eigenvector is given by

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{x}_1(t) = e^{t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is one solution of the system. Next, we need to look for a solution $\mathbf{w}$ of

$$\left(A - \frac{1}{2} I\right) \mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Plugging in for $A - \frac{1}{2}I$, this equation becomes

$$\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix} \mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We see that

$$\mathbf{w} = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix}$$

is a solution of this equation. Therefore,

$$\mathbf{x}_2(t) = te^{t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{t/2} \begin{pmatrix} -2/3 \\ 0 \end{pmatrix}$$

is also a solution of our original system. Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 e^{t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[te^{t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{t/2} \begin{pmatrix} -2/3 \\ 0 \end{pmatrix} \right].$$

The initial condition implies that $c_1 = -2$ and $c_2 = -3/2$. Therefore, the solution of the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} 3 + \frac{3}{2}t \\ -2 - \frac{3}{2}t \end{pmatrix} e^{t/2}.$$