MATH 226

Notes on Assignment 9

3.2 Systems of Two First Order Linear Differential Equations

Practice Problems: 1, 3, 5, 7, 9*, 11*, 15*, 19*, 21, 25, 26, 30*

1. The system is autonomous, nonhomogeneous. Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

3. The system is nonautonomous, homogeneous. Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\mathbf{u}' = \begin{pmatrix} -2t & 1 \\ 3 & -1 \end{pmatrix} \mathbf{u}.$$

5. The system is autonomous, homogeneous. Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

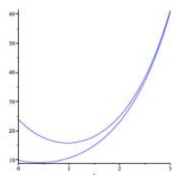
$$\mathbf{u}' = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{u}.$$

7. The system is nonautonomous, nonhomogeneous. Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -2 & \sin t \end{pmatrix} \mathbf{u} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

9.(a) We differentiate: $x' = 3e^t - 7e^{-t} = 2(3e^t + 7e^{-t}) - (3e^t + 21e^{-t}) = 2x - y$ and $y' = 3e^t - 21e^{-t} = 3(3e^t + 7e^{-t}) - 2(3e^t + 21e^{-t}) = 3x - 2y$. Also, $x(0) = 10$ and $x(0) = 24$.

(b)



11.(a) We differentiate:

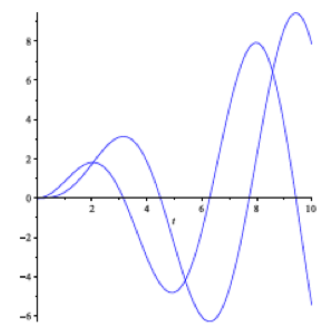
$$\mathbf{x}' = \begin{pmatrix} \cos t - \cos t + t\sin t \\ \sin t + t\cos t \end{pmatrix} = \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}.$$

Also,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2\sin t \end{pmatrix} = \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}.$$

The initial condition is $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b)



15.(a) The system

$$\begin{aligned} x' &= -x + y + 1 = 0 \\ y' &= x + y - 3 = 0 \end{aligned}$$

can be rewritten in matrix form as

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Now

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ thus } A^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

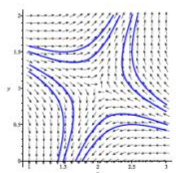
Therefore,

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore, the equilibrium solution is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(b)



(c) Solutions in the vicinity of the critical point tend away from the critical point.

19.(a) The system

$$\begin{aligned} x' &= x + y - 3 = 0 \\ y' &= -x + y + 1 = 0 \end{aligned}$$

can be rewritten in matrix form as

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Now

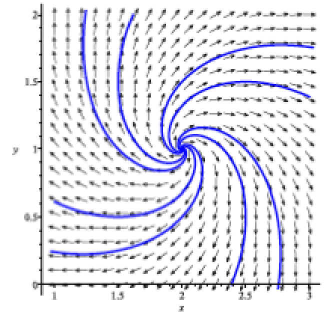
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \text{ thus } A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore, the equilibrium solution is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$



(b)

(c) Solutions in the vicinity of the critical point spiral away from the critical point

21. Let $x_1 = u$ and $x_2 = u'$. Then $x_1' = x_2$ and

$$x_2' = u'' = -2u - 0.5u' = -2x_1 - 0.5x_2.$$

Therefore, we obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5x_2. \end{aligned}$$

25. Let $x_1 = u$ and $x_2 = u'$. Then $x_1' = x_2$ and

$$x_2' = u'' = -0.25u' - 4u + 2\cos 3t = -4x_1 - 0.25x_2 + 2\cos 3t.$$

Now $u(0) = 1$ implies $x_1(0) = 1$ and $u'(0) = -2$ implies $x_2(0) = -2$. Therefore, we obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 - 0.25x_2 + 2\cos 3t \end{aligned}$$

with initial conditions

$$\begin{aligned} x_1(0) &= 1 \\ x_2(0) &= -2. \end{aligned}$$

26. First, divide the equation by t. We arrive at the equation

$$u'' + \frac{1}{t}u' + u = 0.$$

Now let $x_1 = u$ and $x_2 = u'$. Then $x_1' = x_2$ and

$$x_2' = -\frac{1}{t}u' - u = -\frac{1}{t}x_2 - x_1.$$

Now $u(1) = 1$, thus $x_1(1) = u(1) = 1$ and $u'(1) = 0$, thus $x_2(1) = u'(1) = 0$. Therefore, we obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - \frac{1}{t}x_2 \end{aligned}$$

with initial conditions

$$\begin{aligned} x_1(1) &= 1 \\ x_2(1) &= 0. \end{aligned}$$

30.(a) Let $Q_1(t)$ and $Q_2(t)$ be the amount of salt in the respective tanks at time t. Based on conservation of mass, the rate of increase of salt is given by

$$\text{rate of increase} = \text{rate in} - \text{rate out}.$$

For Tank 1, the rate of salt flowing in from Tank 2 is $(Q_2/20) \cdot 1.5 = 0.075Q_2$ ounces/minute. In addition, salt is flowing in from a separate source at the rate of 1.5 ounces/minute. Therefore, the rate of salt flowing in to Tank 1 is $r_{in} = 0.075Q_2 + 1.5$. The rate of flow out of Tank 1 is $r_{out} = (Q_1/30) \cdot 3 = 0.1Q_1$ ounces/minute. Therefore,

$$\frac{dQ_1}{dt} = -0.1Q_1 + 0.075Q_2 + 1.5.$$

Similarly, for Tank 2, salt is flowing in from Tank 1 at the rate of $(Q_1/30) \cdot 3 = 0.1$ oz/min. In addition, salt is flowing in from a separate source at the rate of 3 oz/min. Also, salt is flowing out of Tank 2 at the rate of $4Q_2/20 = 0.2Q_2$ oz/min. Therefore,

$$\frac{dQ_2}{dt} = 0.1Q_1 - 0.2Q_2 + 3.$$

The initial conditions are $Q_1(0) = 55$ and $Q_2(0) = 26$.

(b) This system can be written in matrix form as

$$\mathbf{Q}'(t) = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}.$$

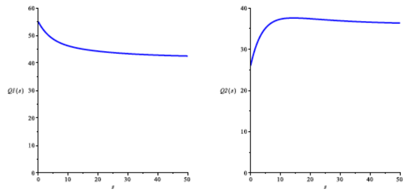
(c) Setting $dQ_1/dt = 0 = dQ_2/dt$, we see that an equilibrium solution must satisfy the system

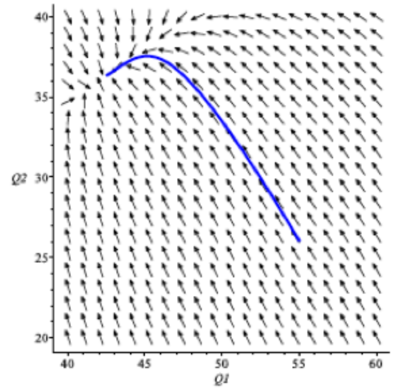
$$\begin{pmatrix} 0.1 & -0.075 \\ -0.1 & 0.2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}.$$

Finding the inverse of the matrix above and multiplying, we find that

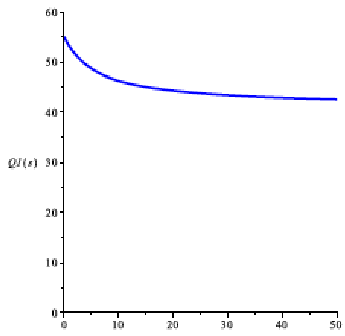
$$\begin{pmatrix} Q_1^E \\ Q_2^E \end{pmatrix} = \frac{1}{0.0125} \begin{pmatrix} 0.525 \\ 0.45 \end{pmatrix} = \begin{pmatrix} 42 \\ 36 \end{pmatrix}.$$

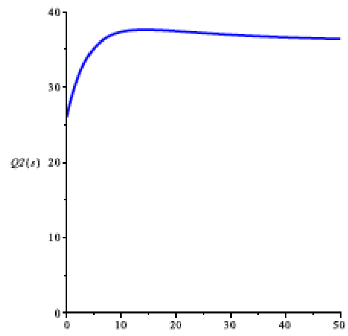
(d)




Q2
Q1
40
35
30
25
20
40
45
50
55
60

(d)


Q1(s)
s
60
50
40
30
20
10
0
0
10
20
30
40
50


Q2(s)
s
40
30
20
10
0
0
10
20
30
40
50