7.5 Periodic Solutions and Limit Cycles

Practice Problems: 1, 3, 8, 10, 11, 15, 16

Feedback Problems: 1, 3, 10, 16

1. The equilibrium solutions of the differential equation are $r = 0$ and $r = 1$. We notice that for $0 < r < 1$, $dr/dt > 0$, while for $r > 1$, $dr/dt < 0$. Therefore, $r = 0$ is an unstable critical point, while $r = 1$ is an asymptotically stable critical point. A limit cycle is given by $r = 1$, $\theta = t + t_0$, which is asymptotically stable.

3. The equilibrium solutions are given by $r = 0$, $r = 2$ and $r = 5$. We notice that $dr/dt > 0$ for $0 < r < 2$ and $r > 5$, while $dr/dt < 0$ for $2 < r < 5$. Therefore, $r = 0$ is an unstable critical point, $r = 2$ is an asymptotically stable critical point, and $r = 5$ is an unstable critical point. A limit cycle is given by $r = 2$, $\theta = t + t_0$, which is asymptotically stable. Another limit cycle is given by $r = 5$, $\theta = t + t_0$. This limit cycle is unstable.

8.(a) Using the fact that

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt},$$

we have

$$r\frac{dr}{dt} = \frac{x^2 f(r)}{r} + \frac{y^2 f(r)}{r} = \frac{r^2 f(r)}{r} = rf(r).$$

Therefore, $r dr/dt = rf(r)$, which implies $dr/dt = f(r)$. Therefore, we have periodic solutions corresponding to the zeros of $f(r)$. To find the direction of motion on the closed trajectories, we use the fact that

$$-r^2\frac{d\theta}{dt} = y\frac{dx}{dt} - x\frac{dy}{dt}.$$

Therefore, for this system, we have

$$-r^2\frac{d\theta}{dt} = y^2 + \frac{xyf(r)}{r} + x^2 - \frac{xyf(r)}{r} = x^2 + y^2 = r^2.$$

Therefore, $d\theta/dt = -1$, which implies $\theta = -t + t_0$. Therefore, the closed trajectories will move in the clockwise direction.

(b) By part (a), we know the periodic solutions will be given by the zeros of f. The zeros are $r = 0, 1, 5, 6$. Using the fact that $dr/dt = f(r)$, we see that $dr/dt > 0$ if $0 < r < 1$ and $r > 5$ and $dr/dt < 0$ if $1 < r < 5$. Therefore, $r = 0$ is unstable, $r = 1$ is asymptotically stable, $r = 5$ is unstable, and $r = 6$ is semistable. We conclude that there is an asymptotically stable limit cycle at $r = 1$ with $\theta = -t + t_0$, an unstable limit cycle at $r = 5$ with $\theta = -t + t_0$ and a semistable periodic solution at $r = 6$ with $\theta = -t + t_0$.

10. Given $F(x,y) = a_{11}x + a_{12}y$ and $G(x,y) = a_{21}x + a_{22}y$, it follows that $F_x + G_y = a_{11} + a_{22}$. Based on the hypothesis, $F_x + G_y$ is either always positive or always negative on the entire plane. By Theorem 7.5.2, the system cannot have a nontrivial periodic solution.

11. Given that $F(x,y) = 4x + y + 3x^3 - y^2$ and $G(x,y) = -x + 5y + x^2y + y^3/3$, $F_x + G_y = 9 + 10x^2 + y^2$ is positive for all (x,y). Therefore, by Theorem 7.5.2, the system cannot have a nontrivial periodic solution.

15.(a) Letting $x = u$ and $y = u'$, we obtain the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + \mu\left(1 - \frac{1}{3}y^2\right)y. \end{aligned}$$

(b) To find the critical points, we need to solve the system

$$\begin{aligned} y &= 0 \\ -x + \mu\left(1 - \frac{1}{3}y^2\right)y &= 0. \end{aligned}$$

We see that the only solution of this system is $(0,0)$. Therefore, the only critical point is $(0,0)$. We notice that this system is almost linear. Therefore, we look at the Jacobian matrix. We see that

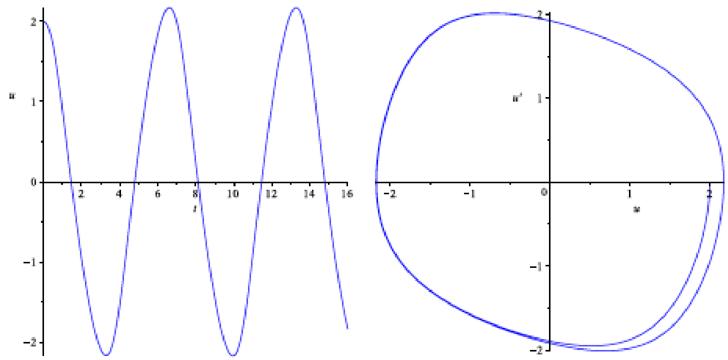
$$\mathbf{J}(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & \mu - \mu y^2 \end{pmatrix}.$$

Therefore,

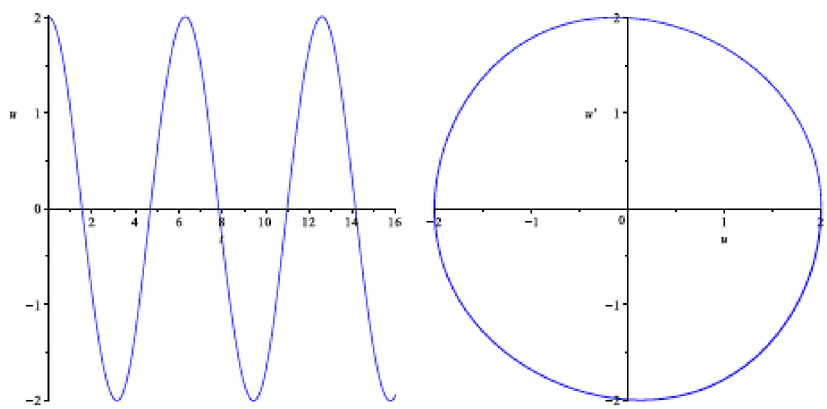
$$\mathbf{J}(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

The eigenvalues are $\lambda = (\mu \pm \sqrt{\mu^2 - 4})/2$. If $\mu = 0$, the equation reduces to the differential equation for the simple harmonic oscillator. In that case, the eigenvalues are purely imaginary and $(0,0)$ is a center, which is stable. If $0 < \mu < 2$, the eigenvalues have non-zero imaginary part with positive real part. In that case, the critical point $(0,0)$ is an unstable spiral. If $\mu > 0$, the eigenvalues are real and both positive. In that case, the origin is an unstable node.

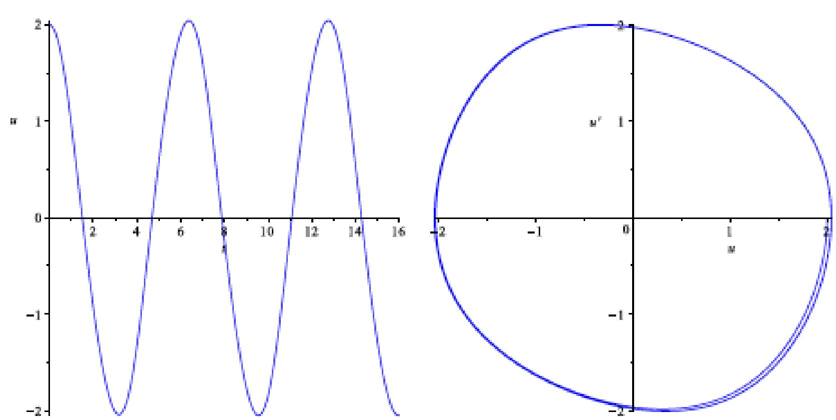
(c) We will consider initial conditions $x(0) = 2$, $y(0) = 0$. For $\mu = 1.0$, $A \approx 2.16$ and $T \approx 6.65$:



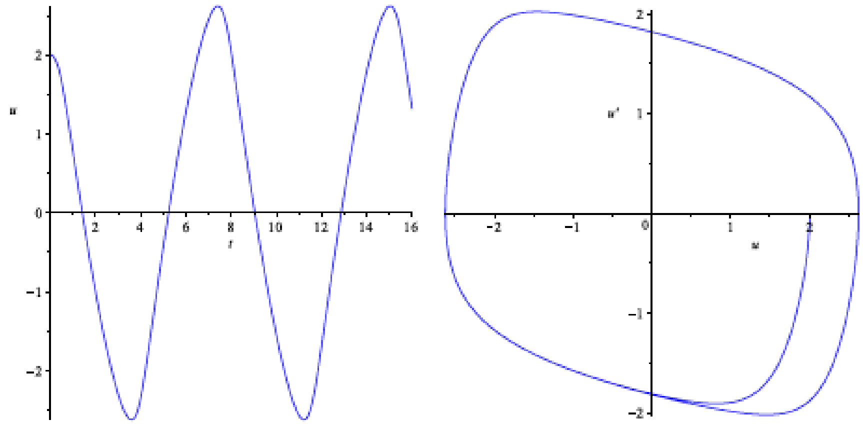
(d) For $\mu = 0.2$, $A \approx 1.99$ and $T \approx 6.31$:



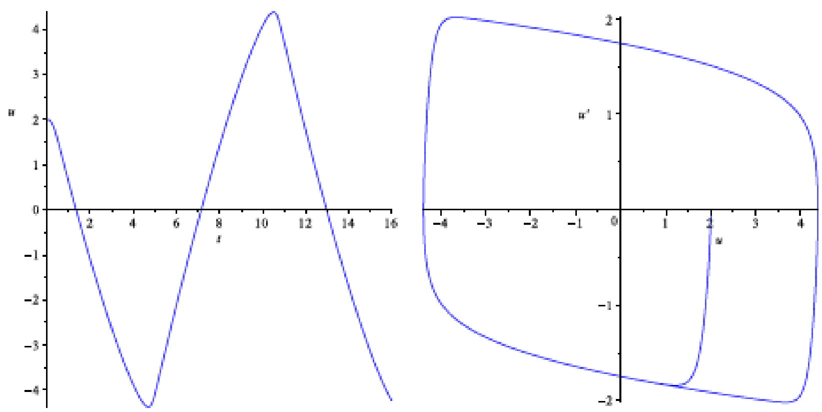
For $\mu = 0.5$, $A \approx 2.03$ and $T \approx 6.39$:



For $\mu = 2.0$, $A \approx 2.60$ and $T \approx 7.65$:



For $\mu = 5.0$, $A \approx 4.36$ and $T \approx 11.60$:



(e)

μ	A	T
0.2	1.99	6.31
0.5	2.03	6.39
1.0	2.16	6.65
2.0	2.60	7.65
5.0	4.36	11.60

16.(a) The critical points are solutions of

$$\begin{aligned} \mu x + y - x(x^2+y^2) &= 0 \\ -x + \mu y - y(x^2+y^2) &= 0. \end{aligned}$$

Multiplying the first equation by y, the second equation by x and subtracting the second equation from the first, we have $x^2 + y^2 = 0$. Therefore, the only critical point is the origin.

(b) The Jacobian matrix is given by

$$\mathbf{J}(x,y) = \begin{pmatrix} \mu - 3x^2 - y^2 & 1 - 2xy \\ -1 - 2xy & \mu - x^2 - 3y^2 \end{pmatrix}.$$

At $(0,0)$,

$$\mathbf{J}(0,0) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}.$$

Thus the linear system near the origin is given by

$$\begin{aligned} x' &= \mu x + y \\ y' &= -x + \mu y. \end{aligned}$$

The eigenvalues for this system are $\lambda = \mu \pm i$. For $\mu < 0$, the origin is a stable spiral. For $\mu = 0$, the origin is a center. For $\mu > 0$, the origin is an unstable spiral.

(c) As usual, let $x = r\cos\theta$ and $y = r\sin\theta$. Then multiplying the first equation of our system by x and the second equation by y, we have

$$\begin{aligned} xx' &= \mu x^2 + xy - x^2(x^2+y^2) \\ yy' &= -xy + \mu y^2 - y^2(x^2+y^2). \end{aligned}$$

Adding these two equations results in

$$xx' + yy' = \mu(x^2+y^2) - (x^2+y^2)^2 = \mu r^2 - r^4.$$

Then, using the fact that

$$xx' + yy' = rr',$$

we conclude that

$$rr' = \mu r^2 - r^4,$$

and thus

$$\frac{dr}{dt} = \mu r - r^3.$$

To find an equation for $d\theta/dt$, we multiply the first equation by y, the second equation by x and subtract the second equation from the first. We conclude that

$$yx' - xy' = x^2 + y^2 = r^2.$$

Using the fact that

$$-r^2\frac{d\theta}{dt} = yx' - xy',$$

we conclude that

$$-r^2\frac{d\theta}{dt} = r^2$$

and then

$$\frac{d\theta}{dt} = -1.$$

(d) Using the equation for dr/dt found in part (c), we see that there is a critical point at $r = 0$ and $r = \sqrt{\mu}$. Further, we note that $dr/dt > 0$ for $0 < r < \sqrt{\mu}$ and $dr/dt < 0$ for $r > \sqrt{\mu}$. Therefore, $r = \sqrt{\mu}$ is an asymptotically stable limit cycle which will attract all nonzero solutions.