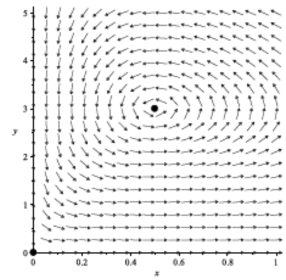
MATH 226: *Notes on Assignment 19*

7.4 Predator-Prey Equations

Practice Problems: 1, 4, 6, 11, 14, 16

Feedback Problems: 1, 6, 11, 14

1.(a)



(b) The critical points are solutions of the system

$$\begin{aligned} x(1.5 - 0.5y) &= 0 \\ y(-0.5 + x) &= 0. \end{aligned}$$

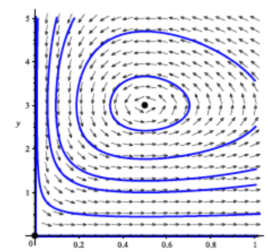
The two critical points are $(0,0)$ and $(0.5, 3)$.

(c) The Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} 3/2 - y/2 & -x/2 \\ y & -1/2 + x \end{pmatrix}.$$

The eigenvalues and eigenvectors are $\lambda_1 = i\sqrt{3}/2$, $\mathbf{v}_1 = (1, -2i\sqrt{3})^T$ and $\lambda_2 = -i\sqrt{3}/2$, $\mathbf{v}_2 = (1, 2i\sqrt{3})^T$. The eigenvalues are purely imaginary. Using the method of Example 1, we get that $(0.5, 3)$ is a center, which is stable.

(d,e)



(f) Except for solutions along the coordinate axes, the other trajectories are closed curves about the critical point $(0.5, 3)$.

At $(0,0)$,

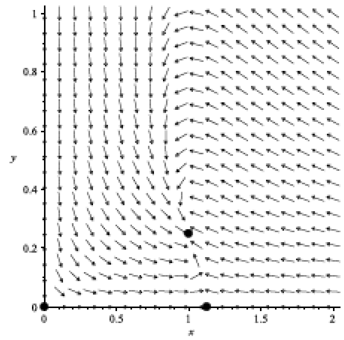
$$\mathbf{J}(0,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are $\lambda_1 = 3/2$, $\mathbf{v}_1 = (1,0)^T$ and $\lambda_2 = -1/2$, $\mathbf{v}_2 = (0,1)^T$. The eigenvalues are of opposite sign. Therefore, $(0,0)$ is a saddle point, which is unstable.

At $(0.5, 3)$,

$$\mathbf{J}(0.5, 3) = \begin{pmatrix} 0 & -1/4 \\ 3 & 0 \end{pmatrix}.$$

4.(a)



(b) The critical points are solutions of the system

$$\begin{aligned} x(1.125 - x - 0.5y) &= 0 \\ y(-1 + x) &= 0. \end{aligned}$$

The three critical points are $(0,0)$, $(9/8,0)$, and $(1,1/4)$.

(c) The Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} 9/8 - 2x - y/2 & -x/2 \\ y & -1 + x \end{pmatrix}.$$

At $(0,0)$,

$$\mathbf{J}(0,0) = \begin{pmatrix} 9/8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are $\lambda_1 = 9/8$, $\mathbf{v}_1 = (1,0)^T$ and $\lambda_2 = -1$, $\mathbf{v}_2 = (0,1)^T$. The eigenvalues are of opposite sign. Therefore, $(0,0)$ is a saddle point, which is unstable.
At $(9/8,0)$,

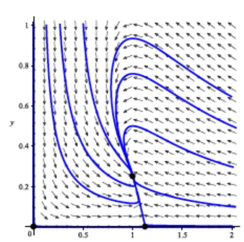
$$\mathbf{J}(9/8,0) = \begin{pmatrix} -9/8 & -9/16 \\ 0 & 1/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are $\lambda_1 = -9/8$, $\mathbf{v}_1 = (1,0)^T$ and $\lambda_2 = 1/8$, $\mathbf{v}_2 = (9,-20)^T$. The eigenvalues have opposite sign. Therefore, $(9/8,0)$ is a saddle point, which is unstable.
At $(1,1/4)$,

$$\mathbf{J}(1,1/4) = \begin{pmatrix} -1 & -1/2 \\ 1/4 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are $\lambda_1 = (-2+\sqrt{2})/4$, $\mathbf{v}_1 = (-2+\sqrt{2},1)^T$ and $\lambda_2 = (-2-\sqrt{2})/4$, $\mathbf{v}_2 = (-2-\sqrt{2},1)^T$. The eigenvalues are both negative. Therefore, $(1,1/4)$ is a stable node, which is asymptotically stable.

(d,e)

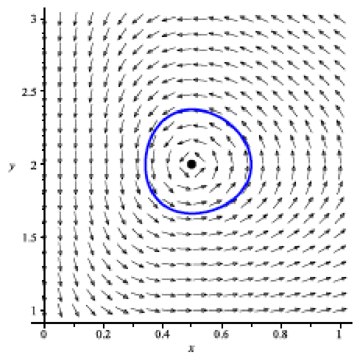


(f) Except for solutions along the coordinate axes, the other trajectories converge to the critical point $(1,1/4)$.

6. Given that t is measured from the time x is a maximum, we have

$$\begin{aligned} x &= \frac{c}{\gamma} + \frac{cK}{\gamma}\cos(\sqrt{ac}t) \\ y &= \frac{a}{\alpha} + K\frac{a}{\alpha}\sqrt{\frac{c}{\alpha}}\sin(\sqrt{ac}t). \end{aligned}$$

The period of oscillation is $T = 2\pi/\sqrt{ac}$. Based on the properties of $\cos(x)$ and $\sin(x)$, we see that the prey population, x is at maximum at $t = 0$ and $t = T$. It reaches a minimum at $t = T/2$. Its rate of increase is greatest at $t = 3T/4$. The rate of the decrease of the prey population is greatest at $t = T/4$. The predator population, y, is maximum at $t = T/4$. It is a minimum at $t = 3T/4$. The rate of increase of the predator population is greatest at $t = 0$ and $t = T$. The rate of decrease of the predator population is greatest at $t = T/2$. In what follows, we consider problem 2 with an initial condition of $x(0) = 0.7$ and $y(0) = 2$. The critical point of interest is $(0.5, 2)$. Since $a = 1$ and $c = 1/4$, the period of oscillation is $T = 4\pi$.



11.(a) Looking at the equation for $x' = 0$, we need $x = 0$ or $\sigma x + 0.5y = 0$ Looking at the equation for $y' = 0$, we need $y = 0$ or $x = 3$. Therefore, the critical points are given by $(0, 0)$, $(1/\sigma, 0)$, and $(3, 2 - 6\sigma)$. As σ increases from zero, the critical point $(1/\sigma, 0)$ approaches the origin, and the critical point $(3, 2-6\sigma)$ will eventually leave the first quadrant and enter the fourth quadrant.

(b) Here, we have $F(x, y) = x(1 - \sigma x - 0.5y)$ and $G(x, y) = y(-0.75 + 0.25x)$. Therefore, the Jacobian matrix for this system is

$$\mathbf{J}(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 - 2\sigma x - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}.$$

We will look at the linear systems near the critical points above. Near the critical point $(0, 0)$, the Jacobian matrix is

$$\mathbf{J}(0,0) = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3/4 \end{pmatrix}$$

and the corresponding linear system near $(0, 0)$ is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3/4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Near the critical point $(1/\sigma, 0)$, the Jacobian matrix is

$$\mathbf{J}(1/\sigma,0) = \begin{pmatrix} F_x(1/\sigma,0) & F_y(1/\sigma,0) \\ G_x(1/\sigma,0) & G_y(1/\sigma,0) \end{pmatrix} = \begin{pmatrix} -1 & -1/2\sigma \\ 0 & (1-3\sigma)/4\sigma \end{pmatrix}$$

and the corresponding linear system near $(1/\sigma, 0)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1/2\sigma \\ 0 & (1-3\sigma)/4\sigma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 1/\sigma$ and $v = y$. Near the critical point $(3, 2 - 6\sigma)$, the Jacobian matrix is

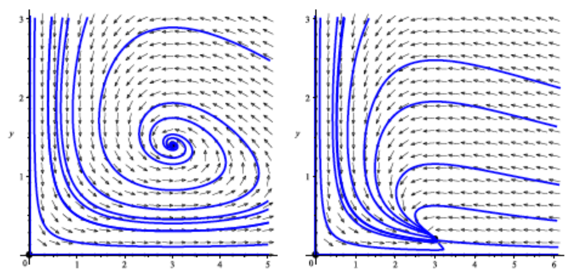
$$\mathbf{J}(3,2-6\sigma) = \begin{pmatrix} F_x(3,2-6\sigma) & F_y(3,2-6\sigma) \\ G_x(3,2-6\sigma) & G_y(3,2-6\sigma) \end{pmatrix} = \begin{pmatrix} -3\sigma & -3/2 \\ 1/2 - 3\sigma/2 & 0 \end{pmatrix}$$

and the corresponding linear system near $(3, 2 - 6\sigma)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -3\sigma & -3/2 \\ 1/2 - 3\sigma/2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 3$ and $v = y - 2 + 6\sigma$. The eigenvalues for the linearized system near $(0,0)$ are given by $\lambda = 1, -3/4$. Therefore, $(0,0)$ is a saddle point. The eigenvalues for the linearized system near $(1/\sigma, 0)$ are given by $\lambda = -1, (1-3\sigma)/(4\sigma)$. For $\sigma < 1/3$, there will be one positive eigenvalue and one negative eigenvalue. In this case, $(1/\sigma, 0)$ will be a saddle point. For $\sigma > 1/3$, both eigenvalues will be negative, in which case $(1/\sigma, 0)$ will be an asymptotically stable node. The eigenvalues for the linearized system near $(3, 2-6\sigma)$ are $\lambda = (-3\sigma \pm \sqrt{9\sigma^2 + 9\sigma - 3})/2$. Solving the polynomial equation $9\sigma^2 + 9\sigma - 3 = 0$, we see that the eigenvalues will have non-zero imaginary part if $0 < \sigma < (\sqrt{21} - 3)/6$. In this case, since the real part, -3σ will be negative, the point $(3, 2-6\sigma)$ will be an asymptotically stable spiral point. If $\sigma > (\sqrt{21} - 3)/6$, then the eigenvalues will both be real. We just need to determine whether they will have the same sign or opposite signs. Solving the equation $-3\sigma + \sqrt{9\sigma^2 + 9\sigma - 3} = 0$, we see that the cut-off is $\sigma = 1/3$. In particular, we conclude that if $(\sqrt{21} - 3)/6 < \sigma < 1/3$, then this critical point will have two real-valued eigenvalues which are negative, in which case this critical point will be an asymptotically stable node. If $\sigma > 1/3$, however, the eigenvalues will be real-valued, but with opposite signs, in which case $(3, 2-6\sigma)$ will be a saddle point. We see that the critical point $(3, 2-6\sigma)$ is the critical point in the first quadrant if $0 < \sigma < 1/3$. From the analysis above, we see that the nature of the critical point changes at $\sigma_1 = (\sqrt{21} - 3)/6$. In particular, at this value of σ, the critical point switches from an asymptotically stable spiral point to an asymptotically stable node.

(c) The two phase portraits below are shown $\sigma = 0.1$ and $\sigma = 0.3$, respectively.



(d) As σ increases, the spiral behavior disappears. For smaller values of σ, the number of prey will decrease, causing a decrease in the number of predators, but then triggering an increase in the number of prey and eventually an increase in the number of predators. This cycle will continue to repeat as the system approaches the equilibrium point. As the value for σ increases, the cycling behavior between the predators and prey goes away.

14.(a) If the prey are harvested, then there will be less prey available for the predators, thus causing a decrease in the number of predators and allowing more of the prey that are not harvested to survive. (As we will see below, the number of prey will not change.) If the predators are harvested, then there will be less predators to eat the prey, thus, allowing the number of prey to increase. As a result, with more prey available, a larger percentage of the predators which are not harvested will be able to survive. (As we will see below the total

number of predators will remain the same.) If they are both harvested, then initially there will be less prey available for the predators, causing a decrease in the number of predators, thus leading to an increase in the number of prey which survive.

(b) The equilibrium solution will occur when $x' = 0$, $y' = 0$. Solving these equations, we see that the equilibrium solution (with non-zero amounts of predators and prey) is given by $((c+E_2)/\gamma, (a-E_1)/\alpha)$. Therefore, if $E_1 > 0$ and $E_2 = 0$, then the number of prey stay the same, but the number of predators decreases.

(c) Using the equilibrium solution from part (b), we see that if $E_1 = 0$, $E_2 > 0$, then the number of prey increases, while the number of predators stays the same.

(d) If both $E_1 > 0$ and $E_2 > 0$, then the number of prey increases and the number of predators decreases.

16.(a) Similar analysis as the solution to 14(a).

(b) If $H_2 = 0$, then solving the predator equation $y' = 0$, we see that the number of prey in the equilibrium solution will be $x = 3$. Then, plugging $x = 3$ into the equation for $x' = 0$, we conclude that the number of predators in the equilibrium solution will be $y = 2 - 2H_1/3$. Therefore, if $H_1 > 0$, but $H_2 = 0$, then the number of prey will remain the same, while the number of predators will decrease.

(c) If $H_1 = 0$, then solving the prey equation $x' = 0$, we see that the number of predators in the equilibrium solution will be $y = 2$. Plugging $y = 2$ into the predators equation $y' = 0$, we conclude that the number of prey in the equilibrium solution will be $x = 2H_2 + 3$. Therefore, if $H_1 = 0$ but $H_2 > 0$, then the number of prey will increase, while the number of predators will stay the same.

(d) Now consider $H_1, H_2 > 0$. The equilibrium solution must satisfy $x(1 - 0.5y) - H_1 = 0$ and $y(-0.75 + 0.25x) - H_2 = 0$. In other words, it must satisfy $x(1 - 0.5y) = H_1 > 0$ and $y(-0.75 + 0.25x) = H_2 > 0$. Therefore, from the first equation, we need $x > 0$ and $1 - 0.5y > 0$ or $x < 0$ and $1 - 0.5y < 0$, but we are assuming the number of prey is positive, therefore, we must have $1 - 0.5y > 0$. In other words, $2 > y$. Therefore, the number of predators will decrease from the number of predators in the presence of no harvesting. Similarly, we must have $-0.75 + 0 - 25x > 0$ which implies $x > 3$. Therefore, the number of prey will increase from the number of prey in the presence of no harvesting.