MATH 226: Notes on Assignment 17

Section 6.7 Defective Matrices: 1, 3, 9

Problem 1

$$A = \begin{pmatrix} 4 & -9 \\ 1 & -2 \end{pmatrix} \text{ has characteristic polynomial } \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

so 1 is an eigenvalue of algebraic multiplicity 2. Examining $(A - \lambda I)\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, we have

$\begin{pmatrix} 3 & -9 & | & a \\ 1 & -3 & | & b \end{pmatrix}$ which row reduces to $\begin{pmatrix} 1 & -3 & | & b \\ 0 & 0 & | & a - 3b \end{pmatrix}$ which has a solution if $a - 3b = 0$

Hence geometric multiplicity of $\lambda = 1$ is 1. To get a bona fide eigenvector $\mathbf{v}$, let $a = 0, b = 0$ so we can take $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

To obtain a generalized eigenvector $\mathbf{w}$ with $(A - \lambda I)\mathbf{w} = \mathbf{v}$, we take $a = 3, b = 1$ which yields

$$\begin{pmatrix} 1 & -3 & | & 1 \\ 0 & 0 & | & 0 \end{pmatrix} \text{ so } w_1 = 3w_2 + 1. \text{ We can take } w_2 = 0 \text{ yielding } \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We then have two linearly independent solutions of $\mathbf{X}' = A\mathbf{X}$:

$$e^t\mathbf{v} = e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } te^t\mathbf{v} + e^t\mathbf{w} = te^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{One fundamental matrix is } \begin{pmatrix} 3e^t & 3te^t + e^t \\ e^t & te^t \end{pmatrix}$$

Many other correct answers are possible. In this case,

$$e^{At} = \begin{pmatrix} e^t(3t+1) & -9te^t \\ te^t & (-3t+1)e^t \end{pmatrix}$$

Problem 3

$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$ has characteristic polynomial $\lambda^3 - +6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3$

so 1 is an eigenvalue of algebraic multiplicity 3. Thus $A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$ which row reduces to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ so the geometric multiplicity is 1. For an eigenvector $\mathbf{v}$, we have $v_1 = 0, v_2 = -v3$ so we can choose $\mathbf{v} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. More generally, we have

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & a \\ 2 & -1 & -1 & b \\ -3 & 2 & 2 & c \end{array}\right) \text{ row reducing to } \left(\begin{array}{ccc|c} 1 & 0 & 0 & a+b \\ 0 & 1 & 1 & 2a+b \\ 0 & 0 & 0 & -a+b+c \end{array}\right)$$

To get the first generalized eigenvector $\mathbf{w}$, set $a = 0, b = -1, c =$ (the components of $\mathbf{v}$); This gives $w_1 = -1, w_2 = -1 - w_3$ and free choice of w_3. Let $w_3 = 0$ so $\mathbf{w} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$. To get the second generalized vector $\mathbf{u}$ so $(A - 2I)\mathbf{u} = \mathbf{w}$, set $a = -1, b = -1, c = 0$. This yields $u_1 = -2, u_2 = -3 - u_3$ so we can take $u_3 =)$ and obtain $\mathbf{u} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$. A set of three linearly independent solutions are $e^{2t}\mathbf{v}$, $te^{2t}\mathbf{v} + e^{2t}\mathbf{w}$, and $\frac{t^2}{2}e^{2t}\mathbf{v} + te^{2t}\mathbf{w} + e^{2t}\mathbf{u}$

We can write a fundamental matrix as $e^{2t}\begin{pmatrix} \mathbf{v} & t\mathbf{v} + \mathbf{w} & \frac{t^2}{2}\mathbf{v} + t\mathbf{w} + \mathbf{u} \end{pmatrix}$

or, in expanded form,

$$e^{2t}\begin{pmatrix} 0 & 0t + -1 & 0\frac{t^2}{2} - t - 2 \\ -1 & -t - 1 & -\frac{t^2}{2} - t - 3 \\ 1 & t & \frac{t^2}{2} + 0t + 0 \end{pmatrix} = e^{2t}\begin{pmatrix} 0 & -1 & -t - 2 \\ -1 & -t - 1 & -\frac{t^2}{2} - t - 3 \\ 1 & t & \frac{t^2}{2} \end{pmatrix}$$

Other correct fundamental matrices are possible. In this case, the matrix exponential is

$$e^{At}\begin{pmatrix} -e^{2t}(t-1) & te^{2t} & te^{2t} \\ \frac{-te^{2t}(t-4)}{2} & \frac{e^{2t}(t^2-2t+2)}{2} & \frac{te^{2t}(t-2)}{2} \\ \frac{te^{2t}(t-6)}{2} & \frac{-te^{2t}(t-4)}{2} & \frac{-e^{2t}(t^2-4t-2)}{2} \end{pmatrix}$$

Problem 9

$$A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \text{ has characteristic polynomial } \lambda^2 - +6\lambda + 9 = (\lambda+3)^2$$

so -3 is an eigenvalue of algebraic multiplicity 2. Examining $(A - \lambda I)\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, we have

$\left(\begin{array}{cc|c} 4 & -4 & a \\ 4 & -4 & b \end{array}\right)$ which row reduces to $\left(\begin{array}{cc|c} 4 & -4 & a \\ 0 & 0 & b-a \end{array}\right)$ which has a solution if $a- = b$

Hence geometric multiplicity of $\lambda = -3$ is 1. To get a bona fide eigenvector $\mathbf{v}$, let $a = 0, b = 0$ so we can take $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

To obtain a generalized eigenvector $\mathbf{w}$ with $(A - \lambda I)\mathbf{w} = \mathbf{v}$, we take $a = 1, b = 1$ which yields

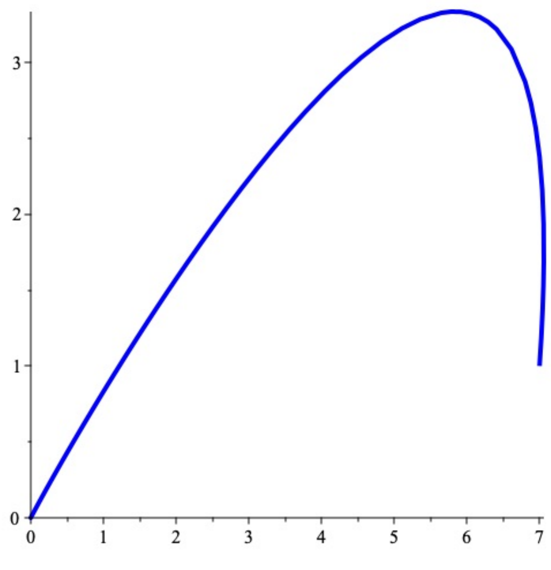
$$\left(\begin{array}{cc|c} 4 & -4 & 1 \\ 0 & 0 & 0 \end{array}\right) \text{ so } 4w_1 = 4w_2 + 1. \text{ We can take } w_2 = 0 \text{ yielding } \mathbf{w} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

We then have two linearly independent solutions of $\mathbf{X}' = A\mathbf{X}$:

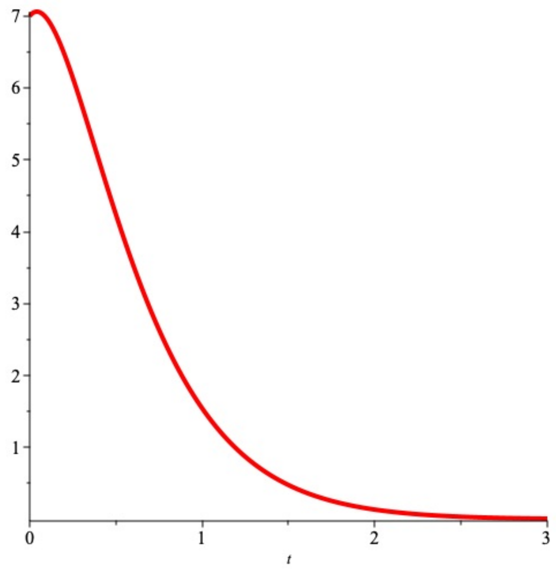
$$e^{-3t}\mathbf{v} = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } te^{-3t}\mathbf{v} + e^{-3t}\mathbf{w} = te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

The general solution is $C_1 e^{-3t}\mathbf{v} + C_2\left(te^{-3t}\mathbf{v} + e^{-3t}\mathbf{w}\right)$ whose value at $t = 0$ is $C_1\mathbf{v} + C_2\mathbf{w}$
To make the value at 0 equal to $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$, we need $1C_1 + (1/4)C_2 = 7, 1C_1 + 0C_2 = 1$. Thus $C_1 = 1$ and $C_2 = 24$. The solution is

$$e^{-3t}\begin{pmatrix} 1 + 24t + 24(1/4) \\ 1 + 24t + 0 \end{pmatrix} = e^{-3t}\begin{pmatrix} 7 + 24t \\ 1 + 24t \end{pmatrix}$$



Trajectory in x_1, X_2 space



Graph of x_1 vs t

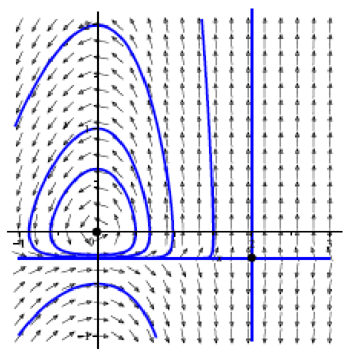
MATH 226: *Notes on Assignment 17*

7.1 Autonomous Systems and Stability.

Practice Problems: 1*, 2*, 4*, 6*, 8*, 23, 24, 25, 26

1.(a) $-2y + xy = 0$ implies $y(-2 + x) = 0$ implies $x = 2$ or $y = 0$. Then, $x + 4xy = 0$ implies $x(1 + 4y) = 0$ implies $x = 0$ or $y = -1/4$. Therefore, the critical points are $(2, -1/4)$ and $(0, 0)$.

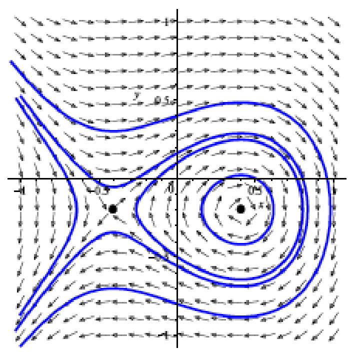
(b)



(c) The critical point $(0, 0)$ is a center, therefore, stable. The critical point $(2, -1/4)$ is a saddle point, therefore, unstable.

2.(a) $1 + 5y = 0$ implies $y = -1/5$. Then, $1 - 6x^2 = 0$ implies $x = \pm 1/\sqrt{6}$. Therefore, the critical points are $(-1/\sqrt{6}, -1/5)$ and $(1/\sqrt{6}, -1/5)$.

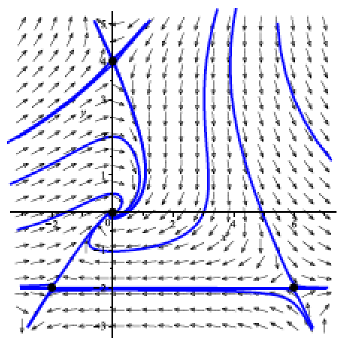
(b)



(c) The critical point $(-1/\sqrt{6}, -1/5)$ is a saddle point, therefore, unstable. The critical point $(1/\sqrt{6}, -1/5)$ is a center, therefore, stable.

4.(a) The equation $-(x-y)(4-x-y)=0$ implies $x-y=0$ or $x+y=4$. The equation $-x(2+y)=0$ implies $x=0$ or $y=-2$. Solving these equations, we have the critical points $(0,0)$, $(0,4)$, $(-2,-2)$, and $(6,-2)$.

(b)

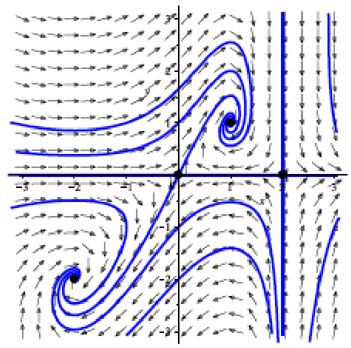


(c) The critical point $(0,0)$ is an asymptotically stable spiral point. The critical point $(0,4)$ is a saddle point, therefore, unstable. The critical point $(-2,-2)$ is a saddle point, therefore, unstable. The critical point $(6,-2)$ is a saddle point, therefore, unstable.

(d) For $(0,0)$, the basin of attraction is bounded below by the line $y=-2$, to the right by a trajectory passing near the point $(5,0)$, to the left by a trajectory heading towards (and then away from) the unstable critical point $(0,4)$, and above by a trajectory heading towards (and then away from) the unstable critical point $(0,4)$.

6.(a) The equation $(2-x)(y-x)=0$ implies $x=2$ or $x=y$. The equation $y(2-x-x^2)=0$ implies $y=0$ or $x=-2$ or $x=1$. The solutions of those two equations are the critical points $(0,0)$, $(2,0)$, $(-2,-2)$, and $(1,1)$.

(b)

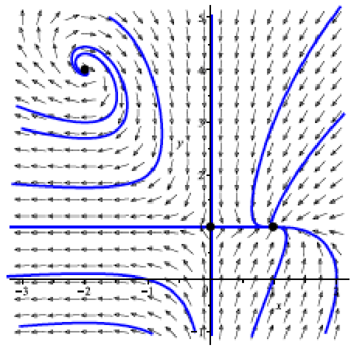


(c) The critical point $(0,0)$ is a saddle point, therefore, unstable. The critical point $(2,0)$ is also a saddle point, therefore, unstable. The critical point $(-2,-2)$ is an asymptotically stable spiral point. The critical point $(1,1)$ is also an asymptotically stable spiral point.

(d) For $(-2,-2)$, the basin of attraction is the region bounded above by the x-axis and to the right by the line $x=2$. For $(1,1)$, the basin of attraction is the region bounded below by the x-axis and to the right by the line $x=2$.

8.(a) The equation $x(2-x-y)=0$ implies $x=0$ or $x+y=2$. The equation $(1-y)(2+x)=0$ implies $y=1$ or $x=-2$. The solutions of those two equations are the critical points $(0,1)$, $(1,1)$, and $(-2,4)$.

(b)



(c) The critical point $(0,1)$ is a saddle point, therefore, unstable. The critical point $(1,1)$ is an asymptotically stable node. The critical point $(-2,4)$ is an unstable spiral point.

(d) For $(1,1)$, the basin of attraction is the right half plane.

23. We compute:

$$\frac{d\Phi}{dt}=\frac{d\phi}{dt}(t-s)=F(\phi(t-s),\psi(t-s))=F(\Phi,\Psi)=F(x,y)$$

and

$$\frac{d\Psi}{dt}=\frac{d\psi}{dt}(t-s)=G(\phi(t-s),\psi(t-s))=F(\Phi,\Psi)=G(x,y).$$

Therefore, $\Phi(t),\Psi(t)$ is a solution for $\alpha+s<t<\beta+s$.

24. Let C_0 be the trajectory generated by the solution $x=\phi_0(t)$, $y=\psi_0(t)$ with $\phi_0(t_0)=x_0$, $\psi_0(t_0)=y_0$ and let C_1 be the trajectory generated by the solution $x=\phi_1(t)$, $y=\psi_1(t)$ with $\phi_1(t_1)=x_0$, $\psi_1(t_1)=y_0$. From problem 23, we know that $\Phi_1(t)=\phi_1(t-(t_0-t_1))$, $\Psi_1(t)=\psi_1(t-(t_0-t_1))$ is a solution. Further, $\Phi_1(t_0)=\phi_1(t_1)=x_0$ and $\Psi_1(t_0)=y_0$. Then, by uniqueness, $\phi_0(t)=\Phi_1(t)$ and $\psi_0(t)=\Phi_1(t)$. Therefore, the trajectories must be the same.

25. If we assume that a trajectory can reach a critical point (x_0,y_0) in a finite length of time, then we would have two trajectories passing through the same point. This contradicts the result in problem 24.

26. Since the trajectory is closed, there is at least one point (x_0,y_0) such that $\phi(t_0)=x_0$, $\psi(t_0)=y_0$ and a number $T>0$ such that $\phi(t_0+T)=x_0$, $\psi(t_0+T)=y_0$. From problem 23, we know that $\Phi(t)=\psi(t+T)$, $\Psi(t)=\psi(t+T)$ will also be a solution. But, then by uniqueness $\Phi(t)=\phi(t)$ and $\Psi(t)=\psi(t)$ for all t. Therefore, $\phi(t+T)=\phi(t)$ and $\psi(t+T)=\psi(t)$ for all t. Therefore, the solution is periodic with period T.