MATH 226: *Notes on Assignment 11*

3.4 Complex Eigenvalues

Practice Problems: 1*, 3*, 5*, 8*, 13*, 15*

1. $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ implies det(A – λI) = $\lambda^2 - 2\lambda + 5$.

Therefore, the eigenvalues are given by $\lambda = 1 \pm 2i$. Now, $\lambda_1 = 1 + 2i$ implies

$$A - \lambda_1 I = \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

is an eigenvector for λ_1 and

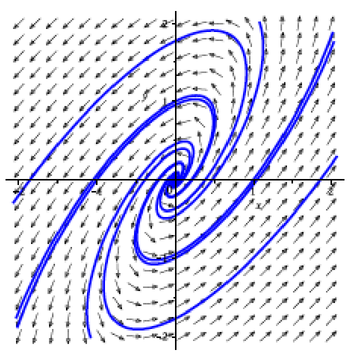
$$\mathbf{x}_1(t) = e^{(1+2i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

is a solution of our system. Further, we use the fact that the real and imaginary parts of $\mathbf{x}_1(t)$ are linearly independent solutions of our system. Now

$$\begin{aligned} \text{Re}\, \mathbf{x}_1(t) &= e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t \right] = e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} \\ \text{Im}\, \mathbf{x}_1(t) &= e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t \right] = e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}. \end{aligned}$$

Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}.$$



The equilibrium is an unstable spiral point.

3.

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

implies

$$\det(A - \lambda I) = \lambda^2 + 1.$$

Therefore, the eigenvalues are given by $\lambda = \pm i$. Now, $\lambda_1 = i$ implies

$$A - \lambda_1 I = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

is an eigenvector for λ_1 and

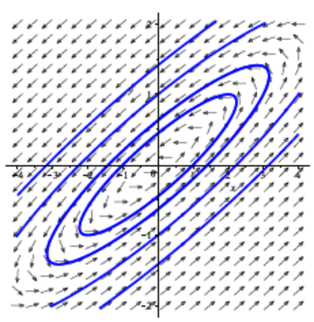
$$\mathbf{x}_1(t) = e^{it} \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

is a solution of our system. Further, we use the fact that the real and imaginary parts of $\mathbf{x}_1(t)$ are linearly independent solutions of our system. Now

$$\begin{aligned} \text{Re } \mathbf{x}_1(t) &= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} \\ \text{Im } \mathbf{x}_1(t) &= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t \right] = \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}. \end{aligned}$$

Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$$



The equilibrium is a stable center.

5.

$$A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$$

implies

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 2.$$

Therefore, the eigenvalues are given by $\lambda = -1 \pm i$. Now, $\lambda_1 = -1 - i$ has corresponding eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$$

is an eigenvector for λ_1 and

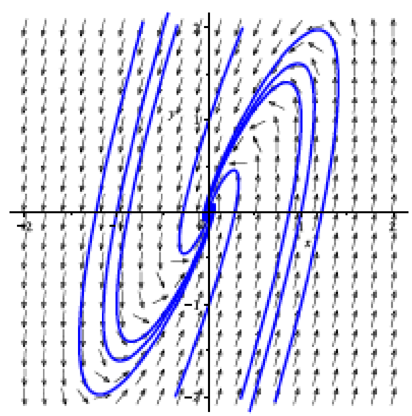
$$\mathbf{x}_1(t) = e^{(-1-i)t} \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$$

is a solution of our system. Further, we use the fact that the real and imaginary parts of $\mathbf{x}_1(t)$ are linearly independent solutions of our system. Now

$$\begin{aligned} \text{Re } \mathbf{x}_1(t) &= e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} \\ \text{Im } \mathbf{x}_1(t) &= e^{-t} \begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}. \end{aligned}$$

Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$



The equilibrium is an asymptotically stable spiral point.

8.

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

implies

$$\det(A - \lambda I) = \lambda^2 + 1.$$

Therefore, the eigenvalues are given by $\lambda = \pm i$. Now, $\lambda_1 = i$ implies

$$A - \lambda_1 I = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

is an eigenvector for λ_1 and

$$\mathbf{x}_1(t) = e^{it} \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

is a solution of our system. Further, we use the fact that the real and imaginary parts of $\mathbf{x}_1(t)$ are linearly independent solutions of our system. Now

$$\begin{aligned} \text{Re } \mathbf{x}_1(t) &= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} \\ \text{Im } \mathbf{x}_1(t) &= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t \right] = \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}. \end{aligned}$$

Therefore, the general solution is given by

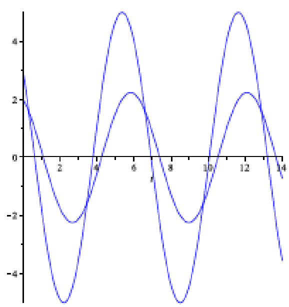
$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$$

The initial condition, $\mathbf{x}(0) = \begin{pmatrix} 3 & 2 \end{pmatrix}^T$ implies

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $c_1 = 2$ and $c_2 = -1$. Thus, the solution is given by

$$\mathbf{x}(t) = 2 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} 3\cos t - 4\sin t \\ 2\cos t - \sin t \end{pmatrix}.$$

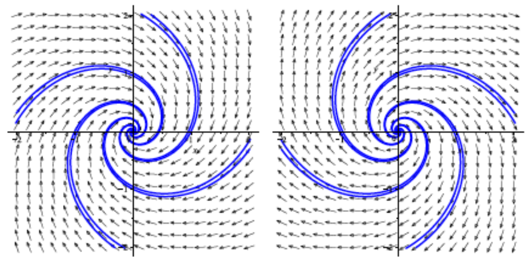


Both components oscillate as $t \to \infty$.

13.(a) The characteristic equation is $\lambda^2 - 2\alpha\lambda + 1 + \alpha^2 = 0$. Therefore, the eigenvalues are $\lambda = \alpha \pm i$.

(b) For $\alpha > 0$, the equilibrium will be an unstable spiral. For $\alpha < 0$, the equilibrium with be a stable spiral. For $\alpha = 0$, the equilibrium will be a center.

(c) Below we show phase portraits for $\alpha = -1/2$ and $\alpha = 1/2$.



15.(a) The characteristic equation is $\lambda^2 + 5\alpha - 4 = 0$. Therefore, the eigenvalues are $\lambda = \pm\sqrt{4 - 5\alpha}$.

(b) If $4 - 5\alpha < 0$, then the eigenvalues are purely imaginary. In that case, the equilibrium point is a center. If $4 - 5\alpha > 0$, then the eigenvalues are real and distinct. In that case, the equilibrium point is a center. Therefore, the critical value for α is $\alpha = 4/5$.

(c) Below we show phase portraits for $\alpha = 2/5$, and $\alpha = 6/5$.