MATH 226 Fall 2022

Group Project 1 (also known as Assignment 7).

Due: Friday, October 7

This first project deals with sustainable management of a renewable resource. It is an important concept in a number of area of the economy including fisheries, forests, and the whaling industry. The discipline that investigates this problem is known as *mathematical bioeconomics*. The creation, investigation and use of the models combines the understanding of mathematics, biology, and economics.

The project examines in part the earliest mathematical models developed in this field. Parts 1 and 2 below can be found on pages 110 and 111 of our text.

I will be forming teams of 3 or 4 students. **Let me know 9 PM Tuesday evening names of any specific students you want on your team. I will try to honor your requests, if possible.**

Project 1 Harvesting a Renewable Resource

Suppose that the population y of a certain species of fish (e.g., tuna or halibut) in a given area of the ocean is described by the logistic equation

$$dy/dt = r(1 - y/K)y.$$

If the population is subjected to harvesting at a rate $H(y,t)$ members per unit time, then the harvested population is modeled by the differential equation

$$dy/dt = r(1 - y/K)y - H(y,t). \qquad (1)$$

Although it is desirable to utilize the fish as a food source, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. The following problems explore some of the questions involved in formulating a rational strategy for managing the fishery.

Project 1 PROBLEMS

1. Constant Effort Harvesting. At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by $H(y,t) = Ey$, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. With this choice for $H(y,t)$, Eq. (1) becomes

$$dy/dt = r(1 - y/K)y - Ey. \qquad \text{(i)}$$

This equation is known as the **Schaefer model** after the biologist M. B. Schaefer, who applied it to fish populations.

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.

(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E. The graph of this function is known as the yield–effort curve.

(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

2. Constant Yield Harvesting. In this problem, we assume that fish are caught at a constant rate h independent of the size of the fish population, that is, the harvesting rate $H(y,t) = h$. Then y satisfies

$$dy/dt = r(1 - y/K)y - h = f(y). \qquad \text{(ii)}$$

The assumption of a constant catch rate h may be reasonable when y is large but becomes less so when y is small.

(a) If $h < rK/4$, show that Eq. (ii) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.

(b) Show that y_1 is unstable and y_2 is asymptotically stable.

(c) From a plot of $f(y)$ versus y, show that if the initial population $y_0 > y_1$, then $y \to y_2$ as $t \to \infty$, but if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.

(d) If $h > rK/4$, show that y decreases to zero as t increases regardless of the value of y_0.

(e) If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable. Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 1(d). The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

3. Use the solution methods we have studied so far to see if they will provide nice closed-form solutions of the differential equations $dy/dt = r(1 - y/K)y - Ey$ and $dy/dt = r(1 - y/k)y - h$.

4. Problems 1 -3 are paper-and-pencil exercises that do not require a computer. For problem 4, you'll investigate how MATLAB can provide additional insight about the behavior of the solutions to the differential equations.

(a) Can the *dsolve* command return solutions to these two equations? If so, sketch the graphs of the solutions.

(b) Select some reasonable numerical values for the constants r, K, E and h and produce direction fields for the associated equations. Can you obtain a reasonable picture of the graphs of the solutions without having formulas for them? For the model in Problem 3, look at examples where r is smaller than, equal to, and greater than $K/4$.