Variation of Parameters Example

Example 1 of Section 6.6

x'= -3x + 4y + sin t, y' = -2x + 3y + t,

x(0) = 0,y(0) = 1

Find Solution of Homogeneous System via Eigenvalues and Eigenvectors

```
syms x s
A = [-3 4; -2 3]
```

```
A = 2×2
    -3     4
    -2     3
```

```
charpoly(A,x)
```

ans = $x^2 - 1$

```
[V,D] = eig(A)
```

```
V = 2×2
   -0.8944   -0.7071
   -0.4472   -0.7071
D = 2×2
    -1     0
     0     1
```

```
X = [ exp(t) 2*exp(-t) ; exp(t) exp(-t)]
```

X =

$$\begin{pmatrix} e^t & 2\,e^{-t} \\ e^t & e^{-t} \end{pmatrix}$$

Compute Matrix Exponential of A

```
X0 = subs(X,t,0)
```

X0 =

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

```
Y = inv(X)
```

Y =

$$\begin{pmatrix} -e^{-t} & 2\,e^{-t} \\ e^t & -e^t \end{pmatrix}$$

```
Y0 = subs(Y,t,0)
```

```
Y0 =
```

$$\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

```
eAt = X*Y0
```

```
eAt =
```

$$\begin{pmatrix} 2\,e^{-t} - e^{t} & 2\,e^{t} - 2\,e^{-t} \\ e^{-t} - e^{t} & 2\,e^{t} - e^{-t} \end{pmatrix}$$

Find Solution of Original Nonhomogeneous System

```
eAt_minus_s = subs(eAt, t, -s)
```

```
eAt_minus_s =
```

$$\begin{pmatrix} 2\,e^{s} - e^{-s} & 2\,e^{-s} - 2\,e^{s} \\ e^{s} - e^{-s} & 2\,e^{-s} - e^{s} \end{pmatrix}$$

```
g = [ sin(t) ; t]
```

```
g =
```

$$\begin{pmatrix} \sin(t) \\ t \end{pmatrix}$$

```
gs = subs(g, t,s)
```

```
gs =
```

$$\begin{pmatrix} \sin(s) \\ s \end{pmatrix}$$

```
Integrand = eAt_minus_s * gs
```

```
Integrand =
```

$$\begin{pmatrix} s\,\left(2\,e^{-s} - 2\,e^{s}\right) - \sin(s)\,\left(e^{-s} - 2\,e^{s}\right) \\ s\,\left(2\,e^{-s} - e^{s}\right) - \sin(s)\,\left(e^{-s} - e^{s}\right) \end{pmatrix}$$

```
IntegrationResult = int(Integrand, s,0,t)
```

```
IntegrationResult =
```

$$\begin{pmatrix} \sigma_1 - 2\,e^{-t}\,(t+1) - 2\,e^{t}\,(t-1) - e^{t}\,(\cos(t) - \sin(t)) + \frac{1}{2} \\ \sigma_1 - 2\,e^{-t}\,(t+1) - e^{t}\,(t-1) - \frac{e^{t}\,(\cos(t) - \sin(t))}{2} + 1 \end{pmatrix}$$

where

$$\sigma_1 = \frac{e^{-t}\,(\cos(t) + \sin(t))}{2}$$

```
xp = eAt * IntegrationResult
```

xp =

$$\begin{pmatrix} (2\,e^{-t} - 2\,e^{t})\,\sigma_2 - (2\,e^{-t} - e^{t})\,\sigma_1 \\ (e^{-t} - 2\,e^{t})\,\sigma_2 - (e^{-t} - e^{t})\,\sigma_1 \end{pmatrix}$$

where

$$\sigma_1 = 2\,e^{t}\ (t-1) + 2\,e^{-t}\ (t+1) - \frac{e^{-t}\ (\cos(t)+\sin(t))}{2} + e^{t}\ (\cos(t)-\sin(t)) - \frac{1}{2}$$

$$\sigma_2 = e^{t}\ (t-1) + 2\,e^{-t}\ (t+1) - \frac{e^{-t}\ (\cos(t)+\sin(t))}{2} + \frac{e^{t}\ (\cos(t)-\sin(t))}{2} - 1$$

```
xparticular = simplify(xp)
```

xparticular =

$$\begin{pmatrix} \frac{3\,e^{t}}{2} - e^{-t} - \frac{\cos(t)}{2} - 4\,t + \frac{3\sin(t)}{2} \\ \frac{3\,e^{t}}{2} - \frac{e^{-t}}{2} - 3\,t + \sin(t) - 1 \end{pmatrix}$$

```
xo = [0;1]
```

xo = 2×1
    0
    1

```
Solution = eAt * xo + xparticular
```

Solution =

$$\begin{pmatrix} \frac{7\,e^{t}}{2} - 3\,e^{-t} - \frac{\cos(t)}{2} - 4\,t + \frac{3\sin(t)}{2} \\ \frac{7\,e^{t}}{2} - \frac{3\,e^{-t}}{2} - 3\,t + \sin(t) - 1 \end{pmatrix}$$