Integration Review

Integrals of a wide variety of functions arise throughout the study of differential equations. This section gathers the main techniques of integration that will be sufficient to handle the integrals encountered in "*Differential Equations: An Introduction to Modern Methods and Applications*," third edition, by James Brannan and William Boyce.

Standard Anti-derivatives

If a function f is continuous on an interval $[a, b]$, then the definite integral $\int_a^b f(x)dx$ exists and is finite. The method of evaluating a definite integral depends on the nature of the integrand $f(x)$. If an anti-derivative $F(x)$ of $f(x)$ can be determined, then the Fundamental Theorem of Calculus states that $\int_a^b f(x)dx = F(b) - F(a)$. As such, having techniques for constructing anti-derivatives is beneficial. The following short list of anti-derivatives of frequently occurring elementary functions is the starting point for the evaluation of more complicated integrals.

$f(x)$	Anti-derivative $F(x)$
$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\tan x \sec x$	$\sec x + C$
a (constant)	$ax + C$
$\dfrac{1}{a^2 + x^2}$ (a is a constant)	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right) + C$
$\dfrac{1}{\sqrt{a^2 - x^2}}$ (a is a constant)	$\arcsin\left(\dfrac{x}{a}\right) + C$
$\sinh x$	$\cosh x + C$
$\cosh x$	$\sinh x + C$
e^x	$e^x + C$

Substitution (Change of Variable)

Objective: To simplify an integral by reducing the integrand to a more easily integrable form, such as one of the entries in the list on the preceding page.

Procedure:

1) Select an expression in the integrand to be called *u*. (See the suggestions for selecting *u* below.)
2) Compute $\dfrac{du}{dx}$ and solve for *du* as though $\dfrac{du}{dx}$ were a fraction, $du = u'(x)dx$.
3) Rewrite the integrand in terms of *u* and *du* by substitution.
4) If variable *u* was well-chosen, then the resulting integrand will be simpler than the original one. If possible, evaluate this integral in terms of *u*.
   Note:
   i) Do not integrate with mixed variables (two different letters).
   ii) The *du* should be in the numerator.
5) Substitute back to obtain the result in terms of the original variable *x*.

Suggestions for selecting u:
1) Try to find an expression to call *u* in the integrand for which the derivative of *u* is essentially also there (in the numerator).
2) Often, such a *u* will be a troublesome or complicated expression (the reduction of which to a single letter will be helpful.)

Example 1. Evaluate $\int x\sqrt{x^2+1}\,dx$

Solution: Let $u = x^2+1$. Then, $\dfrac{du}{dx} = 2x$ so that $\tfrac{1}{2}du = xdx$. Observe that

$$\int x\sqrt{x^2+1}\,dx = \int \sqrt{u}\cdot\frac{1}{2}du = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\frac{2u^{\frac{3}{2}}}{3}+C = \frac{\left(x^2+1\right)^{\frac{3}{2}}}{3}+C.$$

Example 2. Evaluate $\int \dfrac{(\ln x)^3}{3x}dx$

Solution: Let $u = \ln x$. Then, $\dfrac{du}{dx} = \dfrac{1}{x}$ so that $du = \tfrac{1}{x}dx$. Observe that

$$\int \frac{(\ln x)^3}{3x}dx = \int \frac{u^3}{3}du = \frac{u^4}{12}+C = \frac{(\ln x)^4}{12}+C.$$

If your integral has limits you have two choices:
1) Integrate as above resubstituting in terms of x and then plugging in your limits for *x*.
or
2) When you make your substitution into *u*, change your *x* – limits to *u* – limits and evaluate immediately without resubstituting to *x*.

Example 3. Evaluate $\int_0^4 \frac{1}{\sqrt{3x+4}}dx$

Solution:

Way 1. Suppress the limits until the end.

Let $u = 3x+4$. Then, $du = 3dx$ so that $\frac{1}{3}du = dx$. Observe that

$$\int \frac{1}{\sqrt{3x+4}}dx = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-\frac{1}{2}}du = \frac{1}{3}\left(2u^{\frac{1}{2}}\right)+C = \frac{2}{3}\sqrt{3x+4}+C.$$

Now insert the limits:

$$\int_0^4 \frac{1}{\sqrt{3x+4}}dx = \left[\frac{2}{3}\sqrt{3x+4}\right]_0^4 = \frac{2}{3}(\underbrace{\sqrt{3(4)+4}}_{\sqrt{16}=4} - \underbrace{\sqrt{3(0)+4}}_{\sqrt{4}=2}) = \frac{2}{3}(2) = \frac{4}{3}.$$

Way 2. Convert x-limits to u-limits, and proceed with the new definite integral.

Let $u = 3x+4$. Then, $du = 3dx$ so that $\frac{1}{3}du = dx$.

$$\text{When } x = 0,\ u = 3(0)+4 = 4$$
$$\text{When } x = 4,\ u = 3(4)+4 = 16$$

Now, observe that

$$\int_0^4 \frac{1}{\sqrt{3x+4}}\,dx = \frac{1}{3}\int_{u=4}^{u=16}\frac{du}{\sqrt{u}} = \frac{1}{3}\int_4^{16} u^{-\frac{1}{2}}du = \left[\frac{2}{3}u^{\frac{1}{2}}\right]_4^{16} = \frac{2}{3}\left(\sqrt{16}-\sqrt{4}\right) = \frac{2}{3}(4-2) = \frac{2}{3}(2) = \frac{4}{3}$$

INTEGRATION BY PARTS

Objective: To reduce the integral of a product to a simpler form.

When to apply:

1) Check to see if the given integral can easily be integrated by substitution. If it can, use substitution.
2) If not and the integral is a (simple) product, often integration by parts can prove useful.

Formula: There are two common (equivalent) ways of writing the integration by parts formula. In functional form it reads

$$\underbrace{\int \overbrace{f(x)g'(x)}^{\text{One order}}\,dx}_{\text{The integral you seek to find}} = \overbrace{f(x)g(x)}^{\text{Product}} - \underbrace{\int \overbrace{g(x)f'(x)}^{\text{Reverse order}}\,dx}_{\text{Hopefully a simpler integral more amenable to computation}}$$

In traditional form this is often written:

$$\int udv = uv - \int vdu,$$

where to compare with the functional form we have set $u = f(x)$ and $v = g(x)$, so that $dv = g'(x)dx$ and $du = f'(x)dx$.

Procedure:

1) Split up the given integrand to be a product with:

i) One factor called $\begin{Bmatrix} f(x) \\ u \end{Bmatrix}$. This factor should has a $\begin{Bmatrix} derivative\ \ f'(x) \\ differential\ \ du \end{Bmatrix}$.

ii) The other factor is $\begin{Bmatrix} g'(x) \\ dv \end{Bmatrix}$. Integrate this factor to obtain $\begin{Bmatrix} g(x) \\ v \end{Bmatrix}$.

2) Use the parts formula above to obtain an integral on the right side. If this integral is simpler than the original one, well and good. If it is not, then consider whether a different choice of $\begin{Bmatrix} f(x)\ and\ g'(x) \\ u\ and\ dv \end{Bmatrix}$ would be better.

3) Integrate the right side. Keep in mind that sometimes integration by parts must be applied several times in succession to evaluate the original integral.

4) Plug in the limits if you have any.

Example 4: Evaluate $\int xe^{-x}dx$

Solution: Let $f(x) = x$, $f'(x) = 1$ (Simple) and $g'(x) = e^{-x}$, $g(x) = -e^{-x}$ (Easily integrated). Using the integration by parts formula then yields

$$\begin{aligned}
\int xe^{-x}dx &= \underbrace{x}_{f(x)} \cdot \underbrace{(-e^{-x})}_{g(x)} - \int \underbrace{(-e^{-x})}_{g(x)} \cdot \underbrace{(1)}_{f'(x)}\, dx \\
&= -xe^{-x} + \underbrace{\int e^{-x}dx}_{\substack{\text{Note this integral is a} \\ \text{simpler integral than} \\ \text{the original one.}}} \\
&= -xe^{-x} - e^{-x} + C \\
&= -e^{-x}(x+1) + C
\end{aligned}$$

Note that if you had chosen instead $f(x) = e^{-x}$ and $g'(x) = x$, then $f'(x) = -e^{-x}$ (not bad), $g(x) = \dfrac{x^2}{2}$, <u>but</u> then

$$\int xe^{-x}dx = -\frac{x^2}{2}e^{-x} + \underbrace{\int \frac{x^2}{2}e^{-x}dx}_{\substack{\text{This integral is worse than} \\ \text{the original integral.}}}$$

Thus the formula is all right, but the resulting integral is harder than the one you started with. Hence, if you had made this choice of $f(x)$ and $g(x)$, you would go back and make another more judicious selection.

Example 5. Suppose $n \neq -1$. Evaluate $\int x^n \ln x dx$.

Solution: Let

$$u = \ln x,\ du = \tfrac{1}{x}dx,\ dv = x^n dx,\ v = \frac{x^{n+1}}{n+1}.$$

Using the integration by parts formula then yields

$$
\begin{aligned}
\int x^n \ln x dx &= \underbrace{\frac{x^{n+1}}{n+1}}_{v} \cdot \underbrace{\ln x}_{u} - \int \underbrace{\frac{x^{n+1}}{n+1}}_{v} \cdot \underbrace{\frac{1}{x} dx}_{du} \\
&= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n dx \\
&= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \frac{x^{n+1}}{n+1} + C \\
&= \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C
\end{aligned}
$$

Example 6. Evaluate $\int x^2 e^{-x} dx$.

Solution: Sometimes, the integration by parts formula must be applied multiple times in succession. Let $u = x^2$, $du = 2x dx$, $dv = e^{-x} dx$, $v = -e^{-x}$. Using the integration by parts formula then yields

$$
\begin{aligned}
\int x^2 e^{-x} dx &= -x^2 e^{-x} + 2 \underbrace{\int x e^{-x} dx}_{\text{Use parts again.}} \\
&= -x^2 e^{-x} + 2 \underbrace{\left[-x e^{-x} + \int e^{-x} dx \right]}_{\text{See Example 4.}} \\
&= -x^2 e^{-x} + 2 \left[-x e^{-x} - e^{-x} \right] + C \\
&= -e^{-x} \left(x^2 + 2x + 2 \right) + C
\end{aligned}
$$

If there are limits on the integral when applying parts, evaluate these limits in the natural way.

Example 7. Evaluate $\int_1^{e^2} \sqrt{x} \ln x dx$.

Solution: Let $u = \ln x$, $du = \frac{1}{x} dx$, $dv = \sqrt{x}\ dx$, $v = \dfrac{2}{3} x^{\frac{3}{2}}$. (See Example 5.) Using the integration by parts formula then yields

$$
\begin{aligned}
\int_1^{e^2} \sqrt{x} \ln x dx &= \left[\frac{2}{3} x^{\frac{3}{2}} \ln x \right]_1^{e^2} - \frac{2}{3} \int_1^{e^2} \underbrace{x^{\frac{3}{2}} \cdot \frac{1}{x}}_{x^{\frac{1}{2}}} dx \\
&= \left(\frac{2}{3} e^3 \underbrace{\ln e}_{1} - \frac{2}{3} \underbrace{\ln 1}_{0} \right) - \frac{2}{3} \cdot \frac{2}{3} \left[x^{\frac{3}{2}} \right]_1^{e^2} \\
&= \frac{2}{3} e^3 - \frac{4}{9} \left(e^3 - 1 \right) \\
&= \frac{2}{9} e^3 + \frac{4}{9} \\
&= \frac{2}{9} \left(e^3 + 2 \right)
\end{aligned}
$$

Occasionally when applying parts with trigonometric functions, repeated use of the formula results in a cycling back to the original integral. In such case, transpose, combine, and solve for the desired integral as in the following example.

Example 8. Evaluate $\int e^x \sin x dx$.

Solution: Let $u = e^x$, $du = e^x dx$, $dv = \sin x dx$, $v = -\cos x$. Using the integration by parts formula then yields

$$\int e^x \sin x dx = -e^x \cos x + \underbrace{\int e^x \cos x dx}_{\substack{\text{Apply parts in the same} \\ \text{manner as the first application}}}$$

For the integral on the right side, let $u = e^x$, $du = e^x dx$, $dv = \cos x dx$, $v = \sin x$. Observe that

$$\int e^x \sin x dx = -e^x \cos x + \left[e^x \sin x - \underbrace{\int e^x \sin x dx}_{\substack{\text{The original integral} \\ \text{back again}}} + C \right]$$

Solving by transposing all $\int e^x \sin x dx$ to the left yields

$$2\int e^x \sin x dx = e^x (\sin x - \cos x) + C,$$

so that

$$\int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

Integrating Trigonometric Functions

The following trigonometric definitions and identities are often useful when integrating trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

The following formulas can be obtained by using the substitution technique.

$f(x)$	Anti-derivative $F(x)$
$\tan x$	$-\ln\lvert\cos x\rvert + C$
$\cot x$	$\ln\lvert\sin x\rvert + C$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$
$\csc x$	$\ln\lvert\csc x - \cot x\rvert + C$

Integrating products of powers of sines and cosines $\int \sin^m x \cos^n x dx$

Case 1: At least one of *m* and *n* is odd. (Both could be odd, and one of them could be zero.)

From the odd one split off one factor to use in *du* and apply $\sin^2 x + \cos^2 x = 1$ to the remaining even power. Use substitution *u* = the opposite function.

Example 9. Evaluate $\int \sin^3 x \cos^2 x dx$.

Solution: Split off one of the sine terms and use the identity, as follows.

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \underbrace{\sin^2 x}_{=1-\cos^2 x} \cos^2 x (\sin x dx) \\ &= (1-\cos^2 x)\cos^2 x(\sin x dx) \\ &= \int (\cos^2 x - \cos^4 x)(\sin x dx)\end{aligned}$$

Now, let $u = \cos x$, so that $du = -\sin x dx$. Observe that

$$\begin{aligned}\int \sin^3 x \cos^2 x dx = \ldots &= \int (\cos^2 x - \cos^4 x)(\sin x dx) \\ &= -\int (u^2 - u^4) du \\ &= -\frac{u^3}{3} + \frac{u^5}{5} + C \\ &= -\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + C\end{aligned}$$

Case 2: Both *m* and *n* are even (and one of them could be zero).

Use the half-angle formulae, repeatedly if necessary.

Example 10. Evaluate $\int \sin^2 3x \cos^2 3x dx$.

Solution:

$$\begin{aligned}\int \sin^2 3x \cos^2 3x dx &= \int \left(\frac{1-\cos 6x}{2}\right)\left(\frac{1+\cos 6x}{2}\right) dx \\ &= \frac{1}{4}\int (1-\cos^2 6x) dx \\ &= \frac{1}{4}\int \left(1 - \frac{1+\cos 12x}{2}\right) dx \\ &= \frac{1}{8}\int (1-\cos 12x) dx \\ &= \frac{1}{8}\left(x - \frac{\sin 12x}{12}\right) + C\end{aligned}$$

For combinations of $\begin{cases} \tan^n x,\ \sec^n x,\ \tan^m x \sec^n x \\ \cot^n x,\ \csc^n x,\ \cot^m x \csc^n x \end{cases}$ use $\begin{cases} 1+\tan^2 x + \sec^2 x \\ 1+\cot^2 x = \csc^2 x \end{cases}$ along with parts (and substitution) as needed.

Integrating Rational Functions

Let $n(x)$ and $d(x)$ be polynomials. Consider integrating rational expressions of form $\dfrac{n(x)}{d(x)}$.

Case 1: Degree of $n(x)$ is greater than or equal to the degree of $d(x)$

In such case, long divide to rewrite the function as

$$\frac{n(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)},$$

where the degree of $r(x) <$ the degree of $d(x)$.
See if you can apply a substitution or simplify and split into pieces to integrate. If not, apply the procedure outlined in Case 2 below to integrate $\dfrac{r(x)}{d(x)}$.

Case 2: Degree of $n(x)$ is less than the degree of $d(x)$

We break such rational functions down into a sum of simpler fractions guided by the nature of the factors of the denominator.

To this end, factor $d(x)$ into irreducible polynomials of degree 1 or 2.

Each factor $(ax+b)^n$ leads to a contribution to the overall partial fraction decomposition of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

Each factor $(ax^2+bx+c)^n$ leads to

$$\frac{A_1 x + B_1}{ax^2+bx+c} + \frac{A_2 x + B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2+bx+c)^n}$$

The sum of all such expressions obtained for each factor in the denominator of the original rational expression is called its *partial fraction decomposition*. Consider the following example.

Example 11. Determine the form of the partial fraction decomposition for the rational function

$$\frac{3x^4 - 2x + 5}{x^3(x-2)(x+3)(x^2+1)(x^2+2x+2)^2}$$

Solution.
$$\frac{3x^4-2x+5}{x^3(x-2)(x+3)(x^2+1)(x^2+2x+2)^2}$$ decomposes into

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-2}+\frac{E}{x+3}+\frac{Fx+G}{x^2+1}+\frac{Hx+I}{x^2+2x+2}+\frac{Jx+K}{(x^2+2x+2)^2}$$

where the constants A, …, K must be determined to make the two expressions equal.

Once you have the form of the partial fraction decomposition, determine the coefficients A_i and B_i by solving a linear system obtained by equating corresponding coefficients. Substitute them back into the decomposition and integrate each fraction separately. The integrals that arise can be evaluated using

i) regular power rule along with substitution
ii) substitution to get $\ln$ or Arctan forms

Example 12. Evaluate $\int \frac{x^3-x^2-x+7}{x^2-x-2}dx$.

Solution. Since the degree of the numerator is larger than the degree of the denominator, long divide to obtain

$$\begin{array}{r|l} & x \\ x^2-x-2 & x^3-x^2-x+7 \\ & \underline{x^3-x^2-2x} \\ & x+7 \end{array}$$

So, the original integrand can be written as
$$\frac{x^3-x^2-x+7}{x^2-x-2}=x+\frac{x+7}{x^2-x-2}$$
and so,
$$\int\frac{x^3-x^2-x+7}{x^2-x-2}dx=\int\left(x+\frac{x+7}{x^2-x-2}\right)dx=\int xdx+\int\frac{x+7}{x^2-x-2}dx. \qquad \textbf{(1)}$$

We apply the method of Case 2 to evaluate the second integral in (1). The partial fraction decomposition for the rational expression in this integral is
$$\frac{x+7}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2} \qquad \textbf{(2)}$$
We must determine the constants A and B so that (2) holds. To do so, we formulate a system of linear equations whose unknowns are A and B. Observe that

$$\frac{x+7}{(x+1)(x-2)}=\frac{A(x-2)+B(x+1)}{(x+1)(x-2)}=\frac{(A+B)x+(-2A+B)}{(x+1)(x-2)} \qquad \textbf{(3)}$$

Equating coefficients in the numerator (the denominators are no longer needed here) leads to the system

$$\begin{cases} A+B=1 \\ -2A+B=7 \end{cases}$$

Solving this system yields $A=-2,\ B=3$.

Thus

$$\begin{aligned}
\int \frac{x^3-x^2-x+7}{x^2-x-2}\,dx &= \int\left(x+\frac{x+7}{x^2-x-2}\right)dx \\
&= \int\left(x-\frac{2}{x+1}+\frac{3}{x-2}\right)dx \\
&= \frac{x^2}{2}-2\ln|x+1|+3\ln|x-2|+C \\
&= \frac{x^2}{2}+\ln\frac{|x-2|^3}{(x+1)^2}+C
\end{aligned}$$

Example 13. Evaluate $\int \frac{4x^2+2x+24}{x^3+3x^2+9x+27}dx$.

Solution. Since the degree of the numerator is less than the degree of the denominator, you cannot divide so proceed directly to the partial fraction decomposition. Observe that

$$x^3+3x^2+9x+27 = x^2(x+3)+9(x+3) = (x+3)(x^2+9)$$

so that the partial fraction decomposition for the integrand is

$$\frac{4x^2+2x+24}{(x+3)(x^2+9)} = \frac{A}{x+3}+\frac{Bx+C}{x^2+9} \qquad \textbf{(4)}$$

We must determine the constants *A*, *B*, and *C* so that (4) holds. To do so, we formulate a system of linear equations whose unknowns are *A*, *B*, and *C*. Observe that

$$4x^2+2x+24 = A(x^2+9)+(Bx+C)(x+3) = (A+B)x^2+(3B+C)x+(9A+3C) \qquad \textbf{(5)}$$

Equating coefficients in the numerator (the denominators are no longer needed here) leads to the system

$$\begin{cases} A + B = 4 \\ 3B + C = 2 \\ 9A + 3C = 24 \end{cases}$$

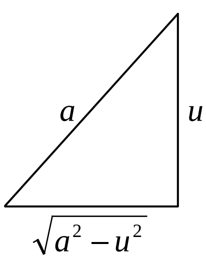
Solving this system yields $A = 3,\ B = 1,\ C = -1$.
Thus,

$$\int \frac{4x^2+2x+24}{x^3+3x^2+9x+27}dx = \int\left(\frac{3}{x+3}+\frac{x-1}{x^2+9}\right)dx$$
$$= 3\int\frac{1}{x+3}dx + \frac{1}{2}\int\frac{2x}{x^2+9}dx - \int\frac{1}{x_2+9}dx$$
$$= 3\ln|x+3| + \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\text{Arctan}\left(\frac{x}{3}\right) + C$$

Trigonometric Substitution

When radicals of the form $\sqrt{a^2-u^2}$ $\sqrt{a^2+u^2}$ $\sqrt{u^2-a^2}$, or even just a^2+u^2 occur, try to form a right triangle with the same radical expression and make the appropriate trig substitution where an angle of the triangle becomes the new variable. More explicitly, see below:

Case $\sqrt{a^2-u^2}$

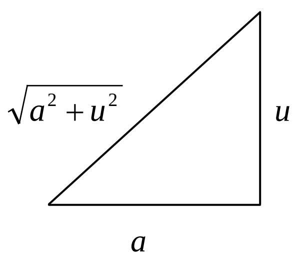


Let $u = a\sin\Theta$

$du = a\cos\Theta d\Theta$

$\sqrt{a^2-u^2} = a\cos\Theta$

Case $\sqrt{a^2+u^2}$ (or a^2+u^2)

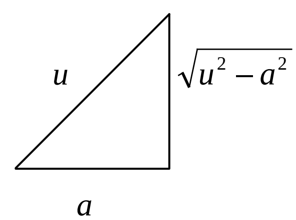


Let $u = a\tan\Theta$

$du = a\sec^2\Theta d\Theta$

$\sqrt{a^2+u^2} = a\sec\Theta$

Case $\sqrt{u^2-a^2}$



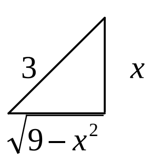
Let $u = a\sec\Theta$

$du = a\sec\Theta\tan\Theta d\Theta$

$\sqrt{u^2-a^2} = a\tan\Theta$

Example 13. Evaluate $\int\frac{x^2dx}{\sqrt{9-x^2}}$.

Solution: Let $x = 3\sin\Theta$ so that $dx = 3\cos\Theta d\Theta$ and $\sqrt{9-x^2} = 3\cos\Theta$. Set up the following right triangle:



Observe that

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{(3\sin\Theta)^2(3\cos\Theta d\Theta)}{3\cos\Theta}$$
$$= 9\int \sin^2\Theta d\Theta$$
$$= 9\int \frac{1-\cos 2\Theta}{2} d\Theta$$
$$= \frac{9}{2}\left(\Theta - \frac{\sin 2\Theta}{2}\right) + C$$

Now we must resubstitute back in terms of x. From our triangle,

$$\frac{x}{3} = \sin\Theta,\ \frac{-\pi}{2} < \Theta < \frac{\pi}{2},$$

so that $\Theta = \text{Arcsin}\left(\frac{x}{3}\right)$. Also

$$\sin 2\Theta = 2\sin\Theta\cos\Theta = 2\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} = \frac{2x}{9}\sqrt{9-x^2}.$$

Hence

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \frac{9}{2}\left(\text{Arcsin}\left(\frac{x}{3}\right) - \frac{x}{9}\sqrt{9-x^2}\right) + C$$
$$= \frac{9}{2}\text{Arcsin}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + C$$

Exercises. Evaluate the following integrals.

1. $\int x e^{2x^2} dx$
2. $\int \frac{ds}{\sqrt{s}\,\left(2+3\sqrt{s}\right)^5}$
3. $\int t^2 \cos\left(1-4t^3\right) dt$
4. $\int x^2 \sin(2x) dx$
5. $\int_1^e \cos\left(\ln x\right) dx$
6. $\int \sin^4 x \cdot \cos^3 x\, dx$
7. $\int \cot(5x) dx$
8. $\int \frac{dx}{\left(x^2+49\right)^{3/2}}$
9. $\int 3x^2\sqrt{3x}\, dx$
10. $\int_1^2 x^{2/5} \ln x\, dx$
11. $\int \frac{x+1}{x^2-4}\, dx$
12. $\int \frac{\csc^2(2x)}{e^{3\cot(2x)}} dx$
13. $\int \frac{1}{x\left(x^2+4\right)}\, dx$
14. $\int \sec^2(\pi x) dx$
15. $\int \cos^2(4x)\, dx$