1. *Mice and Owls Revisited: A Linear Predator Prey Model*. We consider a simple two species ecosystem with mice and owls. The owls only eat mice so if there are no mice, the owl population will decay to zero. The owl is also the only predator for the mice so if there are no owls, the mice will thrive. The presence of owls is bad for the mice while the presence of mice is good for the owls.
Here is a simple model of the Mice and Owl populations, letting x be the population of mice at time t and y the owl population at time t. Suppose our model is the system of differential equations

$$x' = \tfrac{1}{10}x - \tfrac{1}{5}y,\quad y' = \tfrac{3}{10}x - \tfrac{2}{5}y$$

(a) Formulate this model as a linear homogenous system in vector form $\mathbf{x}' = A\mathbf{x}$ by carefully describing the entries of $\mathbf{x}$ and the matrix A.

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \mathbf{x}' = = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, A = \begin{pmatrix} \frac{1}{10} & \frac{-1}{5} \\ \frac{1}{10} & \frac{-2}{5} \end{pmatrix}$$

(b) Find the characteristic polynomial for A.

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} \frac{1}{10} - \lambda & -\frac{1}{5} \\ \frac{3}{10} & -\frac{2}{5} - \lambda \end{pmatrix} = \left(\tfrac{1}{10} - \lambda\right)\left(-\tfrac{2}{5} - \lambda\right) + \tfrac{3}{10} \times \tfrac{1}{5} = \lambda^2 + \frac{3}{10}\lambda + \frac{1}{50}$$

(c) Determine the eigenvalues of A

$$p(\lambda) = \lambda^2 + \frac{3}{10}\lambda + \frac{1}{50} = \left(\lambda + \frac{1}{10}\right)\left(\lambda + \frac{1}{5}\right) \text{ so eigenvalues are } \lambda = -\frac{1}{10} \text{ and } \lambda = -\frac{1}{5}$$

(d) For this model, what will happen to the mice and owl populations in the long term.
Since both eigenvalues are negative, both populations will asymptotically approach 0.

2. The eigenvalues for the matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ are 2 and -3.

(a) For each eigenvalue, find a corresponding eigenvector at least one of whose components is 1.

For $\lambda = 2, A - \lambda I = \begin{pmatrix} 1-2 & 1 \\ 4 & -2-2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}$ so both components of eigenvectors are equal.

We can choose $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = -3, A - \lambda I = \begin{pmatrix} 1-(-3) & 1 \\ 4 & -2-(-3) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$ so any eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ has $v_2 = -4v_1$.

We can choose $\mathbf{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

(b) Find and classify the critical point of the system $\mathbf{x}' = A\mathbf{x}$ for this matrix A:
The origin $\mathbf{0} = (0,0)$ is the only critical point; it is an unstable saddle point because the eigenvalues are of opposite sign.

3. Consider the homogeneous system $\mathbf{x}' = A\mathbf{x}$ where A is the matrix $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

(a) Verify that $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector of this matrix. Without computing a characteristic polynomial, determine the corresponding eigenvalue.

$$A\mathbf{v} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 - 12 \\ 24 - 24 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\mathbf{v} \text{ so eigenvalue is } 0$$

(b) Given that $\lambda = -2$ is another eigenvalue which has $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as an eigenvector, what is the general solution of the system of differential equations?

$$\mathbf{x}(t) = C_1\begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3C_1 + C_2 e^{-2t} \\ 4C_1 + 2C_2 e^{-2t} \end{pmatrix} \text{for any constants } C_1 \text{ and } C_2.$$

(c) Find the particular solution which has $\mathbf{x(0)} = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$

$$\mathbf{x}(0) == \begin{pmatrix} 3C_1 + C_2 e^{0} \\ 4C_1 + 2C_2 e^{0} \end{pmatrix} = \begin{pmatrix} 3C_1 + C_2 \\ 4C_1 + 2C_2 \end{pmatrix} \text{ so we need to solve the system } \begin{matrix} 3C_1 + C_2 = 14 \\ 4C_1 + 2C_2 = 22 \end{matrix}$$

which gives $C_1 = 3$ and $C_2 = 5$.

Thus the particular solution is $x = 9 + 5e^{-2t}, y = 12 + 10e^{-2t}$

4. Let A be the matrix $\begin{pmatrix} -1 & -1/2 \\ 2 & -3 \end{pmatrix}$

(a) Show that A has an eigenvalue $\lambda = -2$ of algebraic multiplicity 2.

$$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & -1/2 \\ 2 & -3-\lambda \end{pmatrix} = (-1-\lambda)(-3-\lambda) - (2)(-1/2) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2.$$

Thus, -2 is a repeated root of algebraic multiplicity 2.

(b) Show that $\lambda = -2$ has geometric multiplicity 1.

$$\text{With } \lambda = -2, A - \lambda I = A + 2I = \begin{pmatrix} -1+2 & -1/2 \\ 2 & -3+2 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 2 & -1 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

since the second row of $A+2I$ is twice the first row. This means each eigenvector is a scalar multiple of $(1,2)^T$; hence the dimension of the solution space of $(A + 2I)\mathbf{v} = \mathbf{0}$ is 1.

(c) Find two linearly independent solutions of the system $\mathbf{x}' = A\mathbf{x}$ for this matrix A.

One solution is $x_1 = e^{-2t}\mathbf{v} = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Another is $x_2 = te^{-2t}\mathbf{v} + e^{-2t}\mathbf{w}$ where $(A-(-2)I)\mathbf{w} = \mathbf{v}$

We must solve $\begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; each solution has $w_1 = 1 + \frac{1}{2}w_2$; we can choose $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Our second solution is $x_2 = te^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-2t}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

5. Suppose Xavier is in love with Yetta, but Yetta is a fickle lover. The more Xavier loves her, the more she dislikes him – but when he loses interest in her, her feelings for him warm up. On the other hand, Xavier reacts to her: when she loves him, his love for Yetta grows and when she loses interest, he also loses interest. Let *x(t)* = Xavier's feelings for Yetta and*y(t)* = Yetta's feelings for Xavier at time *t* where positive and negative values of the variables *x* and *y* denote love and dislike, respectively. The exact units by which these variables can be measured will be left to the imagination of the reader. Our fundamental assumption is: *the change in one person's feelings at any time is directly proportional to the other person's feelings at the same time.*

Suppose we model the relationship between Xavier and Yetta by the system of differential equations

$$dx/dt = ay$$
$$dy/dt = -bx$$

where a and b are positive constants

Determine the general solution of this system and discuss its long term behavior.
We have **x' = Ax** where $A = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$; The characteristic polynomial is det $\begin{pmatrix} \lambda & a \\ -b & \lambda \end{pmatrix} = \lambda^2 + ab$ so the eigenvalues are pure imaginary numbers $\pm i\sqrt{ab}$. For $\lambda = i\sqrt{(ab)}$, the matrix $A - \lambda I$ becomes $\begin{pmatrix} -i\sqrt{ab} & a \\ -b & -i\sqrt{ab} \end{pmatrix}$ so an eigenvector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ has the form $v_2 = \frac{i\sqrt{b}}{\sqrt{a}} v_1$. We can choose $\mathbf{v} = \begin{pmatrix} \sqrt{\boldsymbol{a}} \\ \boldsymbol{i}\sqrt{\boldsymbol{b}} \end{pmatrix}$**.** The general solution is $e^{i\sqrt{ab}\,t}\mathbf{v} = \left(\cos\sqrt{ab}\,t + i\sin\sqrt{ab}\,t\right)\begin{pmatrix} \sqrt{\boldsymbol{a}} \\ \boldsymbol{i}\sqrt{\boldsymbol{b}} \end{pmatrix}$ which yields two real solutions
$x_1 = \begin{pmatrix} \sqrt{a}\cos\sqrt{ab}\,t \\ -\sqrt{b}\sin\sqrt{ab}\,t \end{pmatrix}, x_2 = \begin{pmatrix} \sqrt{a}\sin\sqrt{ab}\,t \\ \sqrt{b}\cos\sqrt{ab}\,t \end{pmatrix}$, General Solution is $C_1x_1 + C_2x_2$

The equilibrium point is the origin and it is the center of a closed orbit; it's actually an ellipse. Xavier and Yetta are trapped in an endless cycle of love and dislike.

6. Let *A* be a 50 by 50 matrix and **0** the function which is the constant zero function; that is, **0*(t)*** is the vector of all zero's for every *t*.

(a) Show that **0** is an equilibrium point for the system of differential equations **x' =Ax**.
*A***0 = 0** and **0' = 0** so **0** is a constant solution to the system.

(b) Prove that if *A* is invertible, then **0** is the only equilibrium point for **x' =*A*x**.
Let **c** be an equilibrium point. Then **c' = 0** so we have A **c = 0.** Since *A* is invertible, we can multiply *A* **c = 0** on the left by A^{-1} to obtain **0** = A^{-1}**0** = A^{-1}(A **c**)= A^{-1}A **c**=I **c** = **c** so **c** must be the zero vector.

(c) Show that if *A* is not invertible, then there is a nonzero equilibrium vector for **x' = Ax**.
If *A* is not invertible, then the algebraic system *A* **c = 0** has nonzero solutions **c** since *A* does not have full rank.

(d) Prove that 0 is an eigenvalue for *A* if and only *A* is not invertible.
A is not invertible if and only if det *A* = 0 if and only if det(*A* – *0*I) = 0 [since *A* – *0I* = *A*] if and only if det (A - λI) = 0 when λ = 0 if and only if 0 is an eigenvalue of *A* [by definition of eigenvalue].