MATH 225 Final Examination

Topics in Linear Algebra and Differential Equations December 14, 2018

1. On December 10, a lone Middlebury College student began circulating a rumor on campus that our president Laurie Patton was going to run for the Democratic nomination for President of the United States in the 2020 election.

The rate at which rumors spread is jointly proportional to the number of people who have heard the rumor and the number who have not.

Suppose x is the number of students who have heard the rumor by time t, and the current enrollment at Middlebury is 2450 students

(a) **[2]** Discuss why the differential equation $x' = k\,x\,(2450 - x)$ with $x(0) = 1$ and k is a positive constant is a reasonable model for the spread of the rumor.

(b) **[3]** How many students will eventually hear the rumor? Justify your answer.

(c) **[4]** Without solving the differential equation, sketch the graph of x as a function of t. Explain why you think your graph is correct.

(d) **[4]** Find the value of x when the rumor is spreading most rapidly.

(e) **[12]** By this morning ($t = 4$), 50 students had heard the rumor. Solve the differential equation and determine the value of k.

2. Consider a predator-prey relationship modeled by the system

$$x' = ax - px^2 - bxy$$
$$y' = -my + nxy$$

Where a, p, b, m and n are positive constants, x is the population of prey and y is the population of predators at time t. The indicated differentiation is, as usual, with respect to t.

(a) **[2]** What happens to the *predator population* (y) in the absence of prey?

(b) **[2]** What happens to the *prey population* (x) in the absence of predators?

Assume for the rest of the problem that $a/p < m/n$.

(c) **[4]** Sketch the curves (*nullclines*) along which $x' = 0$ and $y' = 0$. Then list all the critical points.

(d) **[5]** What is the Jacobian Matrix $J(x,y)$ at an arbitrary point (x,y) in the plane?

(e) **[4]** What is the nature of critical point at the origin?

(f) **[5]** There is another critical point on the positive horizontal axis. Find the signs of the entries of the Jacobian matrix at this point and use them to determine the nature of this critical point.

(g) **[3]** If $x(0)$ and $y(0)$ are both positive, what happens to the populations of the two species as time increases?

3. Part (f) is worth 4 points; each of the other parts has a value of 3 points.

Consider the second order differential equation $t^2 x'' + t x' + x = 0$

(a) **[2]** Show that any linear combinations of solutions of this equation will also be a solution.

(b) **[2]** Show that $f(t) = \cos(\ln |t|)$ and $g(t) = \sin(\ln |t|)$ are solutions of the differential equation.

(c) **[2]** Show that the solutions in (b) form a linearly independent set.

(d) **[2]** Find the particular solution $u(t)$ that satisfies the initial conditions $u(1) = 1$ and $u'(1) = 2$.

(e) **[2]** What is the interval of existence of the solution in (d)?

(f) **[2]** Can there exist three solutions of the differential equation which form a linearly independent set. Why?

(g) **[2]** Rewrite the differential equation as a system of two first order linear differential equations.

(h) **[2]** Find a fundamental solution for the system in (g).

4. (a) **[9]** Consider the nonlinear autonomous system expressed in polar coordinates by the equations

$$\frac{dr}{dt} = r\,|r - 3|\,(r - 1)$$
$$\frac{d\theta}{dt} = +1$$

Determine all periodic solutions, all limit cycles and discuss their stability characteristics.

(b) **[8]**Transform this system to a system of differential equations expressed entirely in Cartesian Coordinates.

(c) **[8]**Show that the system $\begin{aligned}\frac{dx}{dt} &= -2x - 3y - xy^2\\ \frac{dy}{dt} &= y + x^3 - x^2 y\end{aligned}$ has no periodic solutions except the constant solutions.

5. Recall that the *trace* of a matrix is the sum of its main diagonal entries: $a_{11} + a_{22} + \ldots + a_{nn}$ for an $n \times n$ matrix.

Consider the linear system of differential equations $\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x} = \mathbf{Ax}$ where a, b, c, d are real constants. Let T be the trace of A and D be the determinant of A.

(a) **[3]** Find the characteristic polynomial of A in terms of T and D. [For example, you might think (wrongly!) that the characteristic polynomial is $T\lambda^2 + 5(T + D)\lambda - 2018TD$].

(b) **[4]**Find the eigenvalues of A in terms of T and D.

(c) **[18]** For each of the following combinations of signs of T and D, indicate the nature od the eigenvalues and the nature of the critical point at the origin and discuss long-term possibilities for the behavior of the solution of the system:

T	*D*	*Nature of Eigenvalues and Critical Point at Origin*	*Long-Term Behavior of Solution*
0	Positive		
0	Negative		
0	0		
Positive	Negative		
Positive	0		
Negative	Negative		

6. Solve $\mathbf{x}' = \mathbf{Ax}$ where $A = \begin{pmatrix} 2 & -2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 12 \\ 14 \\ 2018 \end{pmatrix}$, today's date.