MATH 226 Sample Examination 1

1. During much of the 20th Century, commercial whaling fleets killed between 20,000 and 30,000 whales per year. Eventually it became apparent that the numbers being killed was putting the whale population at risk. The International Whaling Commission instituted a ban on commercial whaling in 1986. Only three countries remain who disregard the ban: Iceland, Japan and Norway who annually kill 2,000 whales. Left on their own, whales have a natural growth rate of 2.5 percent.

(a) If W is the number of living whales at time t years, discuss why the differential equation

$$W' = .025W - 2000$$

is a reasonable model for studying the dynamics of the whale population

Find the equilibrium solution of the differential equation.

(b) Without solving the differential equation, discuss what will happen to whale population in the long term, according to this model, if the present whale population is

(i) 100,000

(ii) 50,000

(c) Sketch a direction field for this differential equation.

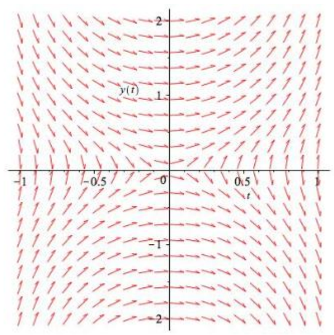
(e) Solve the differential equation by an appropriate change of variable. [Recall that the solution of the differential equation $x' = kx$ is $x = Ce^{kt}$ for some constant C]

(f) Solve the differential equation by the technique of separating variables.

(g) Treat the differential equation as a first order linear one and solve using an integrating factor.

2. Consider the differential equation $y\,y' = x(1+y^2)e^{x^2}$

(a) A student claimed that the direction field for this differential equation looks like this:



(b) What information can you obtain about the nature of solutions of the differential equation (without solving it!) from this direction field ?

(c) Solve the differential equation.

(d) Is the behavior of your solution consistent or inconsistent with the conclusions you derived in part (a)? Give a careful explanation.

3. Existence and Uniqueness of Solutions.
(a) Use the theorem on existence and uniqueness for first order linear differential equations, $y' + p(t)y = g(t)$, (Theorem 2.4.1) to determine (without solving the problem) an interval in which the solution of the initial value problem

$$(4 - t^2)y' + 2ty = 3t^2, \quad y(-3) = 1$$

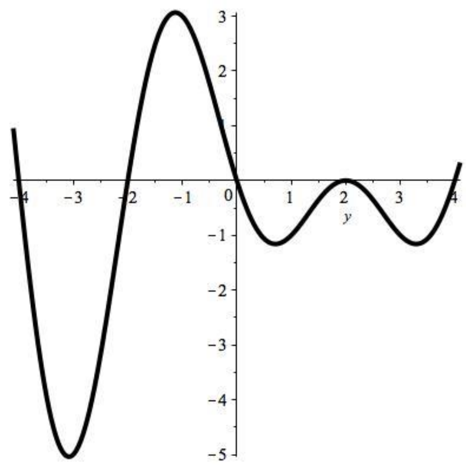
is certain to exist.

(b) Where in the ty-plane are the hypotheses of the theorem about existence and uniqueness (Theorem 2.4.2) for general initial value problem for first order differential equations, $y' = f(t, y), y(t_0) = y_0$, satisfied for the particular problem

$$y' = \sqrt{9 - t^2 - y^2}, y(1) = 2.$$

4. Imagine a medieval world. In this world a Queen wants to poison a King, who has a wine keg with 500 L of his favorite wine. The Queen gives a conspirator a liquid containing 5 g/L of poison, which must be poured slowly into the keg at a rate of 0.5 L/min. The poisoner must also remove the well-stirred mixture at the same rate, so that the keg is not suspiciously full
(a) Formulate a differential equation whose solution would be the amount of poison in the keg at each time *t*. Explain your reasoning and carefully label the units on each term.
(b) Use your equation from (a) to find the amount of poison in the keg at any time, measured from the start of the pouring by the poisoner.
(c) A plot is hatched for the King to drink wine from the keg while he is on a hunt, where he will become so addled that his prey will surely kill him. The poisoner must pour for a time *T*, when the poison in the keg reaches a dangerous concentration of 0.005 g/L. Find *T*.

5. Population Model (Section 2.5) Identify and characterize all of the equilibrium points of the autonomous first order differential equation $y' = f(y)$ where the graph of *f* is shown below:



6. Consider the differential equation $t^2y'' + 5ty' + 4y = 0$, $t > 0$

(a) Show that the sum of any two solutions is a solution.
(b) Show that any constant multiple of a solution is also a solution
(c) What does the result of (a) and (b) tell you about the structure of the set of solutions of this differential equation?
(d) Verify that $y_1(t) = t^{-2}$ is a solution of the differential equation.
(e) Verify that $y_2(t) = t^{-2}\ln t$ is also a solution of the differential equation.