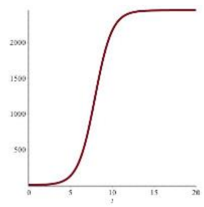
1. (a) $x' = k\,x\,(2450 - x)$.

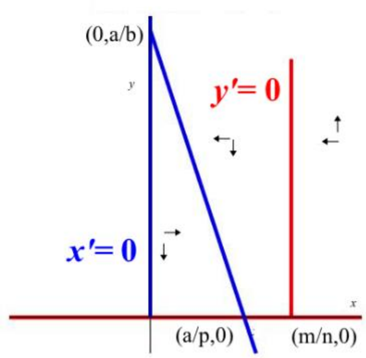
   (b) 2450

   (c) 

   (d) Rumor is spreading most rapidly when x' is maximized. Let $f(x) = k(2450x - x^2)$.
   Since $f'(x) = k(2450 - 2x)$ and $f''(x) = -2k < 0$, the maximum occurs at $x = 1225$.

   (e) Variables separate: $\int \frac{1}{x(2450-x)} dx = \int k\,dt$. Do a partial fraction decomposition of the left hand side to obtain $\int \left(\frac{1}{x} + \frac{1}{2450-x}\right) dx = \int 2450k\,dt$. Integration yields $\ln x - \ln(245 - x) = 2450kt + C$ which we write as $ln\left(\frac{x}{2450-x}\right) = 2450kt + C$. Applying the exponential function to both sides results in $\frac{x}{2450-x} = C\,e^{2450kt}$ With $x(0)$ 1, we obtain $C = \frac{1}{2449}$ Then $\frac{x}{2450-x} = \frac{1}{2449}\,e^{2450kt}$ Solve for x:
   Solution has the form $x(t) = \frac{2450}{1+2449e^{-2450kt}}$. Solving $f(4) = 50$ for k yields $k = \frac{-1}{9800} ln\left(\frac{48}{2449}\right)$
   Or $k = \frac{1}{9800} ln\left(\frac{2449}{48}\right)$

2. Predator-Prey Problem
   (a) Predator population experiences exponential decay; it dies out.
   (b) Prey population experiences logistic growth increasing toward carrying capacity of a/p

   (c)

   (d) $F(x,y) = ax - px^2 - bxy$ so $F_x(x,y) = a - 2px - by$, $F_y(x,y) = -bx$. Also $G(x,y) = -my + nxy$ so $G_x(x,y) = ny$, $G_y(x,y) = -m + nx$.

$$J(x,y) = \begin{bmatrix} a - 2px - by & -bx \\ ny & -m + nx \end{bmatrix}$$

   (e) $J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -m \end{pmatrix}$. Eigenvalues are real and of opposite sign, so origin is a saddle point.

(f) The other critical point is $(a/p,0)$. The Jacobian here is $J\left(\frac{a}{p},0\right) = \begin{pmatrix} -a & -ba/p \\ 0 & n\left(\frac{a}{p} - \frac{m}{n}\right) \end{pmatrix} = \begin{pmatrix} - & - \\ 0 & - \end{pmatrix}$ so both eigenvalues are negative and this is an asymptotically stable point.

   (g) The predator population will die out and the prey population will approach its carrying capacity $\frac{a}{p}$

3. (a) If f and g are any two solutions, then using the fact that the derivative of a sum is the sum of the derivatives, we have $t^2(af+bg)'' + t(af+bg)' + (af+bg) = t^2(af''+bg'')+t(af'+bg')+af+bg = a(t^2f'' + tf' + f) + b(t^2g'' + tg' + g) = a0 + b0 = 0$ for any constants a and b.-

(b) If $f(x) = \cos(\ln|t|)$, then $f'(t) = \frac{-\sin(\ln|t|)}{t}$ and $f''(t) = \frac{t\frac{-\cos(\ln|t|)}{t} + \sin(\ln|t|)}{t^2} = \frac{-\cos(\ln|t|)+\sin(\ln|t|)}{t^2}$

Then $t^2f''(x) + tf'(t) + f(t) = -\cos(\ln|t|) + \sin(\ln|t|) - \sin(\ln|t|) + \cos(\ln|t|) = 0$ so f is a solution. The verification that $\sin(ln|t|)$ is also a solution is similar.

(c) Suppose (*) $C_1\cos(\ln|t|) + C_2\sin(\ln|t|) = 0$ for some constants C_1 and C_2. Then the identity must be true when $t = 1$, but this gives $0 = C_1\cos(\ln|1|) + C_2\sin(\ln|1|) = C_1\times 1 + C_2\times 0 = C_1$. Thus $C_1=0$. Substituting into (*) yields $C_2\sin(\ln|t|) = 0$ but $\sin(\ln|t|)$ is nonzero for some values of t so C_2 must also = 0. Then the pair of functions forms a linearly independent set. [You can also do this problem using the Wronskian.].

$$W = det\begin{pmatrix} \cos(\ln|t|) & \sin(\ln|t|) \\ \frac{-\sin(\ln|t|}{t} & \frac{\cos(\ln|t|)}{t} \end{pmatrix} = \frac{[\cos(\ln|t|]^2 + [\sin(\ln|t|]^2}{t} = \frac{1}{t}$$

(d) $1 = u(1) = C_1$ and $u'(t) = \frac{-C_1\sin(\ln|t|)}{t} + \frac{C_2\cos(\ln|t|)}{t}$ so $2 = u'(1) = C_2$.

Hence $u(t) = \cos(\ln|t|) + 2\sin(\ln|t|)$

(e) Either $-\infty < t < 0$ or $0 < t < \infty$ depending on whether to is negative or positive.

(f) No. The set of solutions of this second order linear homogeneous differential equation is a two dimensional vector space so there can be no linearly independent set of three or more elements.

(g) Let $x_1 = x$ and $x_2 = x'$. Then $x_1' = x_2$ and $x_2' = x'' = -\frac{x}{t^2} - \frac{x'}{t} = -\frac{1}{t^2}x_1 - \frac{1}{t}x_2$. We can write the system as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1/t^2 & -1/t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(h) Let $\mathbf{x} = \begin{pmatrix} cos(ln|t|) \\ \frac{-sin(ln|t|)}{t} \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} sin(ln|t|) \\ \frac{cos(ln|t|)}{t} \end{pmatrix}$. Then {**x, y**} is a fundamental solution.

4. (a) Periodic solutions at $r = 0$ (one point), $r = 1$, and $r = 3$. $r = 0$ is asymptotical stable, $r = 1$ is unstable, $R = 3$ is semistable.

(b) $F_x + G_y \le (-2 - y^2) + (1 - x^2) = -1 - x^2 - y^2 < 0$ is negative throughout the entire (x,y)-plane.

(c) Using $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y & x \\ -x & y \end{pmatrix}\begin{pmatrix} -\theta' \\ r'/r \end{pmatrix} = \begin{pmatrix} y & x \\ -x & y \end{pmatrix}\begin{pmatrix} -1 \\ |r-3|(r-1) \end{pmatrix} =$

$$\begin{pmatrix} -y + x|r-3|(r-1) \\ x + y|r-3|(r-1) \end{pmatrix} \begin{matrix} x' = x|\sqrt{x^2+y^2} - 3|(\sqrt{x^2+y^2} - 1) - y \\ y' = x + y|\sqrt{x^2+y^2} - 3|(\sqrt{x^2+y^2} - 1) \end{matrix}$$

5. (a) $\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - T\lambda + D$

(b) From quadratic formula: $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

T	D	Eigenvalues	
0	Positive	$\pm i\sqrt{D}$ center	Periodic
0	Negative	$\pm\sqrt{D}$ saddle point	
0	0	0 improper node	
Positive	Negative	Opposite signs; Saddle	
Positive	0	T,0	
Negative	Negative	Both Negative	Asymptotically Stable

6. The characteristic polynomial is $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3$so $\lambda = 2$ has algebraic multiplicity 3, but its geometric multiplicity is only 1.

Here $A - \lambda I = A - 2I = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ which gives $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

The format of the solution will vary depending on the choice of an eigenvector and the generalized eigenvectors selected. We want **v, u, w** such that (A - $\lambda I)\mathbf{v} = \mathbf{0}$, (A $-$ $\lambda I)\mathbf{u} = \mathbf{v}$, (A $-$ $\lambda I)\mathbf{w} = \mathbf{u}$. I chose $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 1 \\ -3/4 \\ -1/4 \end{pmatrix}$

which yielded $C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, + C_2\left(te^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix}\right) + C_3\left(\frac{t^2}{2}e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ -3/4 \\ -1/4 \end{pmatrix}\right)$.

At t = 0: C1 + C2 + C3 = 5
0C1 –(1/2)C2 –(3/4)C3 = 14
0C1 + 0C2 –(1/4)(C3)= 2018

$$-4031e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 12108\left(te^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}\right) - 8072\left(\frac{t^2}{2}e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ -3/4 \\ -1/4 \end{pmatrix}\right)$$

In every case, the solutions should reduce to

$$x(t) = (-4036t^2 + 4008t + 5)e^{2t}, y(t) = (4036t + 14)e^{2t}, z(t) = 2018e2^t$$

Alternatively, we can solve the system "bottom up" as three first order linear differential equations:

$z' = 2z, z(0) = 2018$ yields $z = 2018e^{2t}$

$y' = 2y + 2z = 2y + 4036e^{2t}$ or $y' - 2y = 4036e^{2t}$ has Integrating Factor e^{-2t}

which has solution $y = 4036te^{2t} + 14e^{2t}$

$x' = 2x - 2y + 2z$ or $x' - 2x = -8072\, te^{2t} - 28e^{2t}$ also has Integrating Factor e^{-2t}

and has solution $x = (-4036t^2 + 4008t + 5)e^{2t}$