MATH 226: Finding Eigenvectors for Eigenvalues

The procedure for determining an eigenvector associated with a given eigenvalue of a square matrix A is conceptually the same for a genuinely complex eigenvalue (imaginary part not equal to 0) as with a real eigenvalue. We need to find a nontrivial solution to a system of linear algebraic equations. The associated arithmetic only looks more daunting because a number like 3 - $5i$ looks more complicated than a number like 7.

In the treatment below, we simplify the arithmetic by writing the eigenvalue as a single symbol λ as long as possible.

Suppose A is a 2 by 2 matrix with real entries that has a complex number λ as one of its eigenvalues so

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \quad A - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}.$$

Observe first that if b or c are 0, then $\det(A - \lambda I) = (a-\lambda)(d-\lambda)$ so A had 2 **real** eigenvalues a and d. Thus we may assume that both b and c are nonzero.
Note that $A - \lambda I$ has determinant 0 so it is noninvertible and hence the rows are linearly dependent. In solving the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is just the pair of equations $\begin{array}{l} (a-\lambda)v_1 + bv_2 = 0 \\ cv_1 + (d-\lambda)v_2 = 0 \end{array}$
Since the rows of $A - \lambda I$ are linearly dependent, the two equations are equivalent. We can work with either one of them. If we take the first equation, we can solve for v_2 in terms of v_1:

$$bv_2 = -(a-\lambda)v_2 \text{ so } v_2 = \frac{\lambda - a}{b} v_1$$

and choose any nonzero value for v_1, say $v_1 = 1$ to get an eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ \frac{\lambda-a}{b} \end{pmatrix} \text{ or even } \mathbf{v} = \begin{pmatrix} b \\ \lambda - a \end{pmatrix}$$

. Working with the second equation yields an eigenvector of the form

$$\mathbf{v} = \begin{pmatrix} \frac{\lambda-d}{c} \\ 1 \end{pmatrix} \text{ or } \mathbf{v} = \begin{pmatrix} \lambda - d \\ c \end{pmatrix}$$

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<u>Example</u>: Exercise 5 of Section 3.4: Here $A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$ so $a = 1, b = -1$, and $\lambda = -1-i$.
This makes

$$\frac{\lambda - a}{b} = \frac{-1-i-1}{-1} = 2 + i \text{ so } \mathbf{v} = \begin{pmatrix} 1 \\ \frac{\lambda-a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ 2+1 \end{pmatrix}$$