Examples With Complex Eigenvalues

X' = AX

(Note : *Maple* writes *i* as I)

Our Example:

$a := 2; b := \frac{5}{2}; c := -1; d := 3;$

$$a := 2$$

$$b := \frac{5}{2}$$

$$c := -1$$

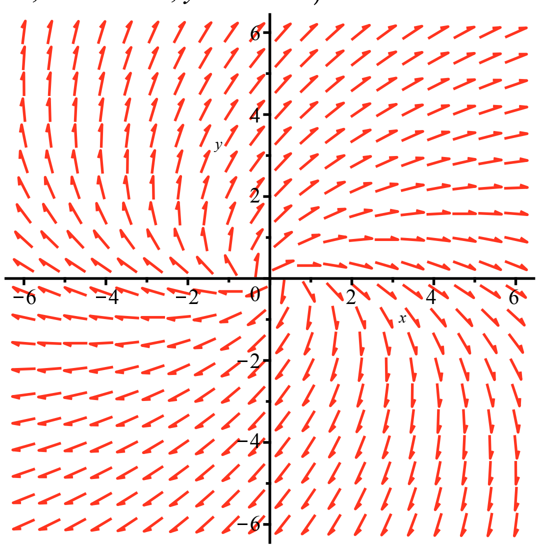
$$d := 3 \quad \textbf{(1)}$$

$ode1 := x'(t) = a \cdot x(t) + b \cdot y(t); ode2 := y'(t) = c \cdot x(t) + d \cdot y(t)$

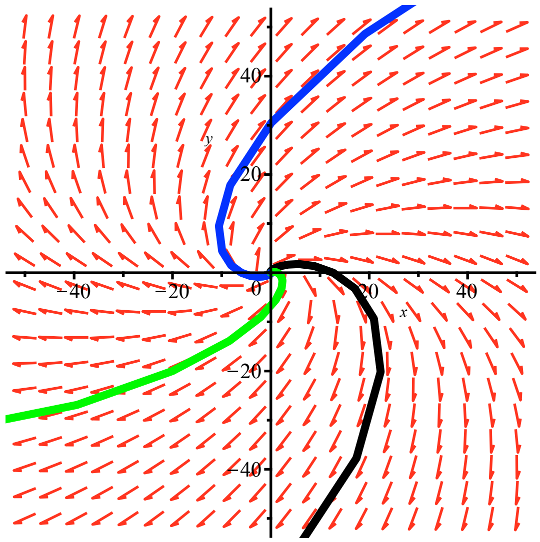
$$ode1 := \mathrm{D}(x)(t) = 2\,x(t) + \frac{5\,y(t)}{2}$$

$$ode2 := \mathrm{D}(y)(t) = -x(t) + 3\,y(t) \quad \textbf{(2)}$$

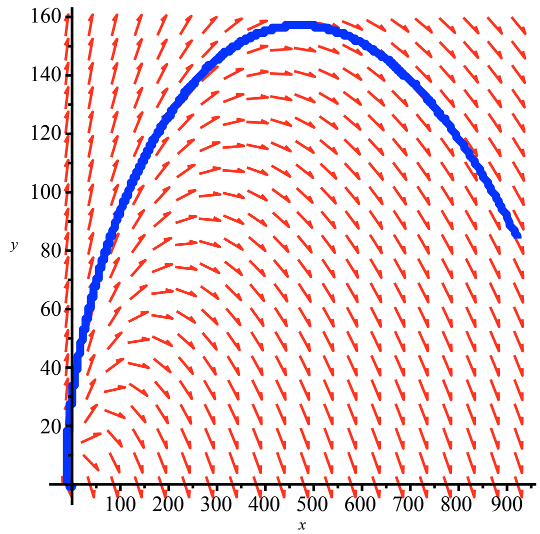
$sys := [ode1, ode2]:$
$with(DEtools):$
$DEplot(sys, [x(t), y(t)], t = 0..2, x = -6..6, y = -6..6)$



$DEplot(sys, [x(t), y(t)], t = -6..2, x = -50..50, y = -50..50, [[x(0) = 1, y(0) = 1], [x(0) = -6, y(0) = 0], [x(0) = -20, y(0) = -20]], linecolor = [black, blue, green])$

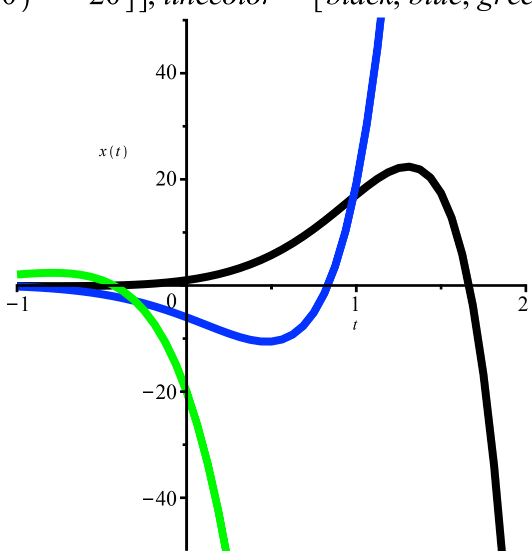


DEplot(*sys*, [*x*(*t*), *y*(*t*)], *t* = −6 ..2 , [[*x*(0) = −6, *y*(0) = 0]], *linecolor* = [*blue*], *numpoints* = 5000)

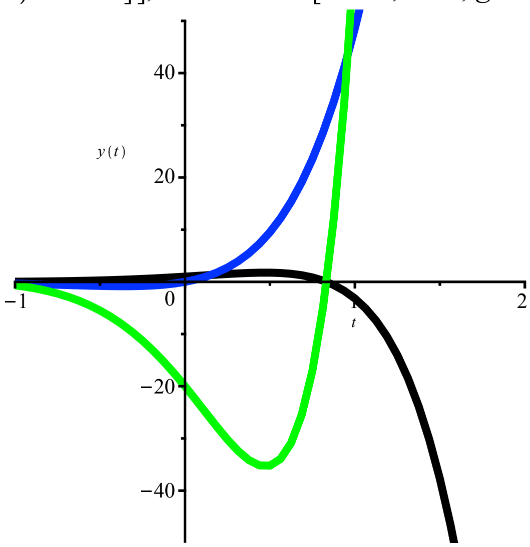


Let's examine the component plots: *x(t)* as a function of *t*, *y(t)* as a function of *t*:

DEplot(*sys*, [*x*(*t*), *y*(*t*)], *t* = −1 ..2, *x* = −50 ..50, *y* = −50 ..50, [[*x*(0) = 1, *y*(0) = 1], [*x*(0) = −6, *y*(0) = 0], [*x*(0) = −20, *y*(0) = −20]], *linecolor* = [*black*, *blue*, *green*], *scene* = [*t*, *x*(*t*)])



$DEplot(sys, [x(t), y(t)], t = -1 ..2, x = -50 ..50, y = -50 ..50, [[x(0) = 1, y(0) = 1], [x(0) = -6, y(0) = 0], [x(0) = -20, y(0) = -20]], linecolor = [black, blue, green], scene = [t, y(t)])$



Another Example With Real Part of Eigenvalue > 0

with(*LinearAlgebra*) :
$a := 3 : b := -2 : c := 4 : d := -1 :$
$A := Matrix([[a, b], [c, d]])$

$$A := \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \tag{3}$$

CharacteristicPolynomial(*A*, lambda)

$$\lambda^2 - 2\lambda + 5 \tag{4}$$

Eigenvalues(*A*)

$$\begin{bmatrix} 1 + 2\,\mathrm{I} \\ 1 - 2\,\mathrm{I} \end{bmatrix} \tag{5}$$

Eigenvectors(*A*)

$$\begin{bmatrix} 1 + 2\,\mathrm{I} \\ 1 - 2\,\mathrm{I} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} + \frac{\mathrm{I}}{2} & \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 & 1 \end{bmatrix} \tag{6}$$

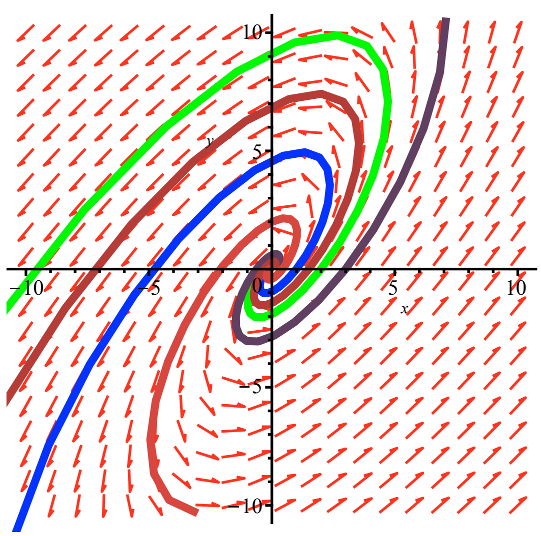
[Note: We could take as an eigenvector **v** with v1 = 1+ *i*, and v2 = 2]
$sys := [x'(t) = a \cdot x(t) + b \cdot y(t), y'(t) = c \cdot x(t) + d \cdot y(t)]$

$$sys := [\mathrm{D}(x)(t) = 3\,x(t) - 2\,y(t), \mathrm{D}(y)(t) = 4\,x(t) - y(t)] \tag{7}$$

dsolve(*sys*)

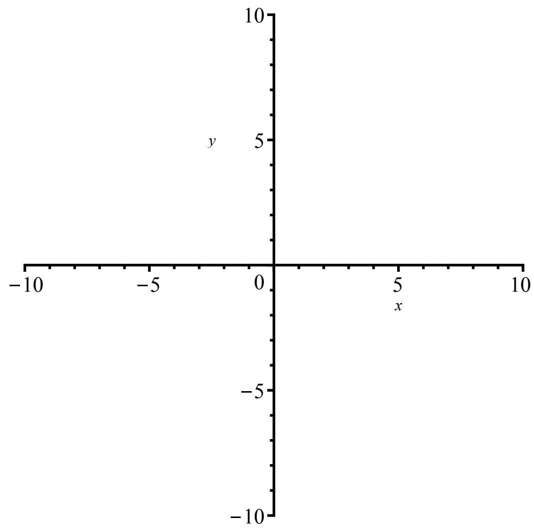
$$\{x(t) = \mathrm{e}^t\,(\sin(2\,t)\,\_C1 + \cos(2\,t)\,\_C2), y(t) = \mathrm{e}^t\,(\sin(2\,t)\,\_C1 + \sin(2\,t)\,\_C2 - \cos(2\,t)\,\_C1 + \cos(2\,t)\,\_C2)\} \tag{8}$$

with(*DEtools*) :
DEplot(*sys*, [*x*(*t*), *y*(*t*)], *t* = −3 ..3, *x* = −10 ..10, *y* = −10 ..10, [[*x*(0) = 1, *y*(0) = 0], [*x*(0) = 2, *y*(0) = 0], [*x*(0) = 3, *y*(0) = 0], [*x*(0) = 1.5, *y*(0) = 0], [*x*(0) = 0, *y*(0) = 2]], *linecolor* = [*blue*, *green*, *violet*, *brown*, *orange*])



We can animate the trajectories

$DEplot(sys, [x(t), y(t)], t = 0..3, x = -10..10, y = -10..10, [[x(0) = 1, y(0) = 0], [x(0) = 2, y(0) = 0], [x(0) = 3, y(0) = 0], [x(0) = 1.5, y(0) = 0], [x(0) = 0, y(0) = 2]], linecolor = [blue, green, violet, brown, orange], arrows = none, animate = true)$



An Example with Pure Imaginary Eigenvalue
(Real Part of Eigenvalue is 0)

$a := 2 : b := -5 : c := 1 : d := -2 :$
$A := Matrix([\,[a, b], [c, d]])$

$$A := \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \tag{9}$$

$CharacteristicPolynomial(A, \text{lambda})$

$$\lambda^2 + 1 \tag{10}$$

$Eigenvalues(A)$

$$\begin{bmatrix} I \\ -I \end{bmatrix} \tag{11}$$

$Eigenvectors(A)$

$$\begin{bmatrix} I \\ -I \end{bmatrix}, \begin{bmatrix} 2+I & 2-I \\ 1 & 1 \end{bmatrix} \tag{12}$$

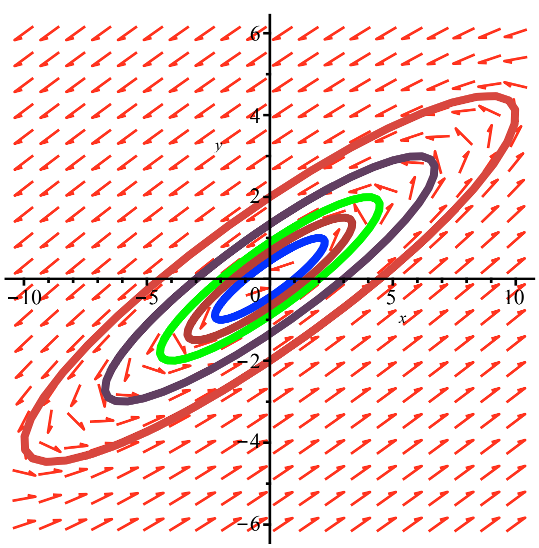
$sys := [\, x'(t) = a\cdot x(t) + b\cdot y(t), y'(t) = c\cdot x(t) + d\cdot y(t)\,]$

$$sys := [\mathrm{D}(x)(t) = 2\,x(t) - 5\,y(t), \mathrm{D}(y)(t) = x(t) - 2\,y(t)] \tag{13}$$

$dsolve(sys)$

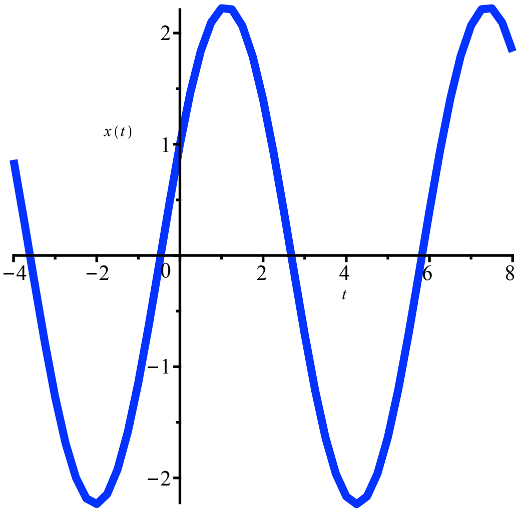
$$\left\{x(t) = \_C1 \sin(t) + \_C2 \cos(t), y(t) = -\frac{\_C1 \cos(t)}{5} + \frac{\_C2 \sin(t)}{5} + \frac{2\,\_C1 \sin(t)}{5} + \frac{2\,\_C2 \cos(t)}{5}\right\} \tag{14}$$

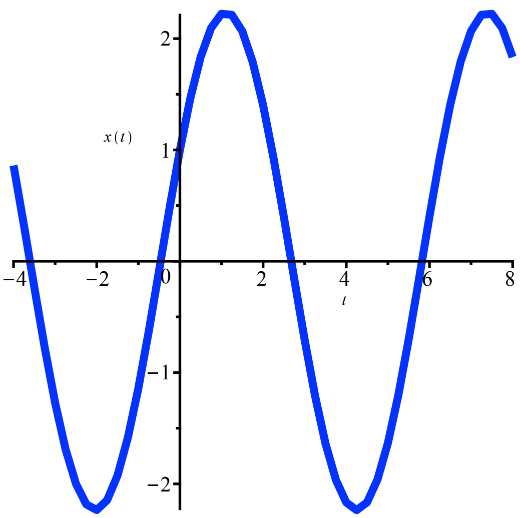
$with(DEtools) :$
$DEplot(sys, [x(t), y(t)], t = -4..4, x = -10..10, y = -6..6, [[x(0) = 1, y(0) = 0], [x(0) = 2, y(0) = 0], [x(0) = 3, y(0) = 0], [x(0) = 1.5, y(0) = 0], [x(0) = 0, y(0) = 2]], linecolor = [blue, green, violet, brown, orange])$



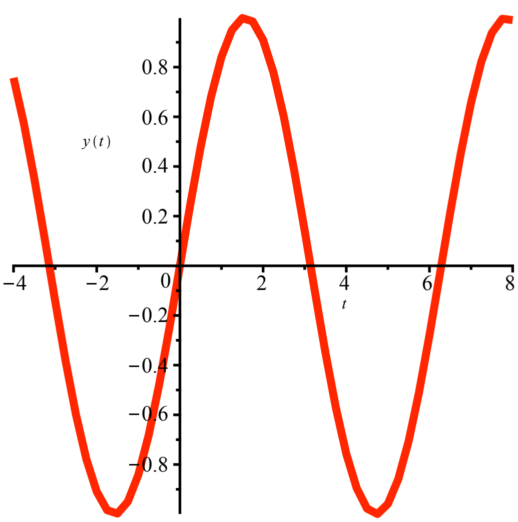
COMPONENT PLOTS

$C1 := DEplot(sys, [x(t), y(t)], t = -4..8, [[x(0) = 1, y(0) = 0]], linecolor = blue, scene = [t, x(t)])$

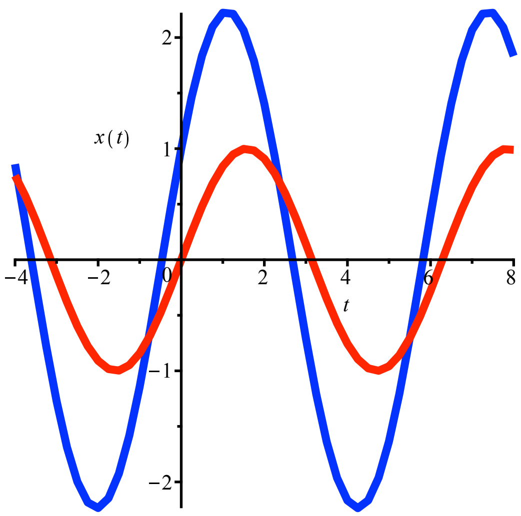




$C2 := DEplot(sys, [x(t), y(t)], t = -4..8, [[x(0) = 1, y(0) = 0]], linecolor = red, scene = [t, y(t)])$



$with(plots):$
$display(C1, C2)$



An Example Where Real Part of Eigenvalue is Negative

$a := 1 : b := -1 : c := 5 : d := -3 :$
$A := Matrix([[a, b], [c, d]])$

$$A := \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \quad (15)$$

$CharacteristicPolynomial(A, \text{lambda})$

$$\lambda^2 + 2\lambda + 2 \quad (16)$$

$Eigenvalues(A)$

$$\begin{bmatrix} -1 + \mathrm{I} \\ -1 - \mathrm{I} \end{bmatrix} \quad (17)$$

$Eigenvectors(A)$

$$\begin{bmatrix} -1 + \mathrm{I} \\ -1 - \mathrm{I} \end{bmatrix}, \begin{bmatrix} \frac{2}{5} + \frac{\mathrm{I}}{5} & \frac{2}{5} - \frac{\mathrm{I}}{5} \\ 1 & 1 \end{bmatrix} \quad (18)$$

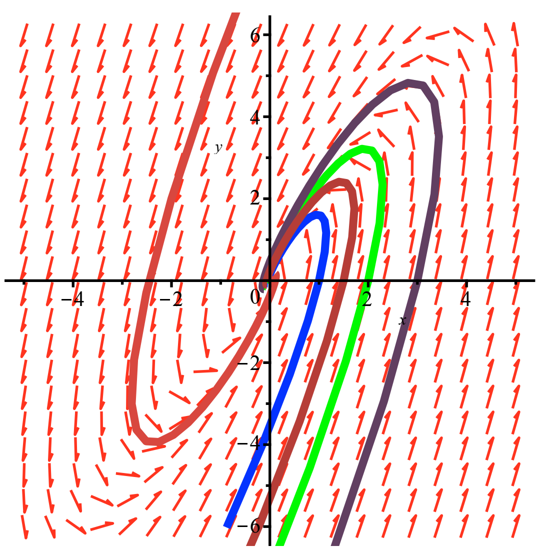
$sys := [x'(t) = a \cdot x(t) + b \cdot y(t), y'(t) = c \cdot x(t) + d \cdot y(t)]$

$$sys := [\mathrm{D}(x)(t) = x(t) - y(t), \mathrm{D}(y)(t) = 5\,x(t) - 3\,y(t)] \quad (19)$$

$dsolve(sys)$

$$\{x(t) = \mathrm{e}^{-t}(\_C1 \sin(t) + \_C2 \cos(t)), y(t) = -\mathrm{e}^{-t}(\_C1 \cos(t) - 2\,\_C2 \cos(t) - 2\,\_C1 \sin(t) - \_C2 \sin(t))\} \quad (20)$$

$with(DEtools):$
$DEplot(sys, [x(t), y(t)], t = -4..4, x = -5..5, y = -6..6, [[x(0) = 1, y(0) = 0], [x(0) = 2, y(0) = 0], [x(0) = 3, y(0) = 0], [x(0) = 1.5, y(0) = 0], [x(0) = -2, y(0) = 2]], linecolor = [blue, green, violet, brown, orange])$



$DEplot(sys, [x(t), y(t)], t = -4..4, x = -5..5, y = -6..6, [[x(0) = 1, y(0) = 0], [x(0) = 2, y(0) = 0],$

$[x(0) = 3, y(0) = 0], [x(0) = 1.5, y(0) = 0], [x(0) = -2, y(0) = 2]], \mathit{linecolor} = [\mathit{blue}, \mathit{green}, \mathit{violet}, \mathit{brown}, \mathit{orange}], \mathit{animate} = \mathit{true}, \mathit{arrows} = \mathit{none})$