MATH 226 Differential Equations

Class 35: December 9, 2022

Handouts

Assignment 23

Announcements

Project 3 Due Today

Course Response Forms
In Class Next Monday
Bring Laptop/SmartPhone

Final Exam
Thursday, December 15: 7 - 10 PM

Numerical Methods for Systems of First Order Equations

Write system

$$\begin{aligned} x' &= F(t, x, y), x(t_0) = x_0 \\ y' &= G(t, x, y), y(t_0) = y_0 \end{aligned}$$

as $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{f} = \begin{pmatrix} F \\ G \end{pmatrix}$$

Euler's Method ($x_{n+1} = x_n + hf_n$) becomes

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n$$

In component form

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h \begin{pmatrix} F(t_n, x_n, y_n) \\ G(t_n, x_n, y_n) \end{pmatrix}$$

Runge-Kutta for Systems

$$\mathbf{x_{n+1}} = \mathbf{x_n} + \frac{h}{6}\left[\mathbf{k_{n1}} = 2\mathbf{k_{n2}} + 2\mathbf{k_{n3}} + \mathbf{k_{n4}}\right]$$

where

$$\begin{aligned}
\mathbf{k_{n1}} &= \mathbf{f}\left(t_n, \mathbf{x_n}\right) \\
\mathbf{k_{n2}} &= \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{x_n} + \frac{h}{2}\mathbf{k_{n1}}\right) \\
\mathbf{k_{n3}} &= \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{x_n} + \frac{h}{2}\mathbf{k_{n2}}\right) \\
\mathbf{k_{n4}} &= \mathbf{f}\left(t_n + h, \mathbf{x_n} + h\mathbf{k_{n3}}\right)
\end{aligned}$$

Example

$$x' = x + y + t, \text{ with } x(0) = 1$$
$$y' = 4x - 2y, \text{ with } y(0) = 0$$

Euler's Method yields approximate values at
$t_n = 0.2, 0.40.6, 0.8.1.0$ with $h = 0.1$ of

$$\begin{pmatrix} 1.26 \\ 0.76 \end{pmatrix}, \begin{pmatrix} 1.7714 \\ 1.4824 \end{pmatrix}, \begin{pmatrix} 2.58991 \\ 2.3703 \end{pmatrix}, \begin{pmatrix} 3.82374 \\ 3.60413 \end{pmatrix}, \begin{pmatrix} 5.64246 \\ 5.38885 \end{pmatrix}$$

Runge-Kutta yields approximate values at
$t_n = 0.2, 0.40.6, 0.8.1.0$ with $h = 0.1$ of

$$\begin{pmatrix} 1.32489 \\ 0.75916 \end{pmatrix}, \begin{pmatrix} 1.9369 \\ 1.57999 \end{pmatrix}, \begin{pmatrix} 2.93459 \\ 2.66201 \end{pmatrix}, \begin{pmatrix} 4.48422 \\ 4.22784 \end{pmatrix}, \begin{pmatrix} 6.8444 \\ 6.56684 \end{pmatrix}$$

Exact Values are

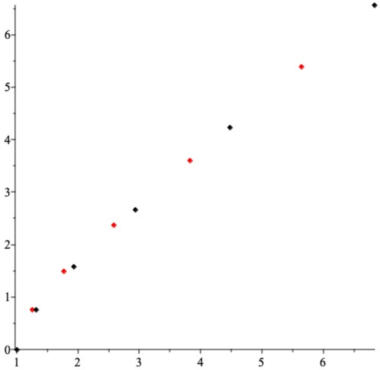
$$\begin{pmatrix} 1.32489 \\ 0.759546 \end{pmatrix}, \begin{pmatrix} 1.93692 \\ 1.58003 \end{pmatrix}, \begin{pmatrix} 2.93463 \\ 2.66208 \end{pmatrix}, \begin{pmatrix} 4.48430 \\ 4.22795 \end{pmatrix}, \begin{pmatrix} 6.84457 \\ 6.84457 \end{pmatrix}$$

$$x' = x + y + t, y' = 4x - 2y \text{ with } x(0) = 1, y(0) = 0$$

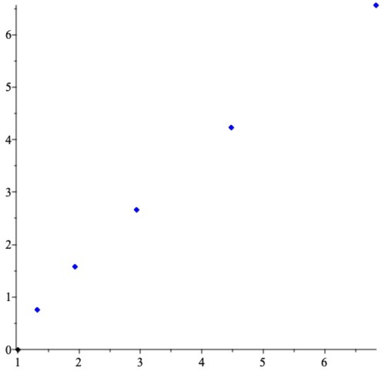
Exact Solution is

$$x(t) = e^{2t} + \frac{2}{9}e^{-3t} - \frac{t}{3} - \frac{2}{9}$$

$$y(t) = e^{2t}. - \frac{8}{9}e^{-3t} = \frac{2t}{3} - \frac{1}{9}$$



Euler vs Exact



Runge-Kutta vs Exact

Ways To Study Differential Equations

- Numerical Approximations to Solution
- Linearize Near Equilibrium Points
- Qualitative Analysis
- **SOLVE THE EQUATION**

Example

$y' = y$ with $y(0) = 1$

$$y'(t) = y(t)$$

$$\frac{y'(t)}{y(t)} = 1$$

Integrate Both Sides With Respect to t

$$\ln y = t + C$$

Exponentiate

$$y = Ce^t$$

Use Initial Condition: $y = 1$ when $t = 0$

$$1 = Ce^0 = C \times 1 = C$$

$y = e^t$ is the unique solution

$$e^t$$

How Do We Actually Calculate e^t for a specific t?
In particular, what is the value of $e^1 = e$?

What Is Going On Inside The Calculator?

Power Series Approach

Assume solution can be represented by a power series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = \sum_{n=0}^{\infty} a_n x^n$$

Then we have an expression for the solution if we can determine what the constant coeffients $a_0, a_1, a_2, ..$ are.

Example: Solve $y' = y$ with initial value $y(0) = 1$
With

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + .a_4 x^4 + a_5 x^5 ...$$

we have

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + ...$$

Since $y' = y$ and

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + .a_4 x^4 + a_5 x^5 ...$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + ...$$

we can equate coefficients

$$a_1 = a_0 \quad 2a_2 = a_1 \quad 3a_3 = a_2 \quad \dots \quad na_n = a_{n-1}$$
$$a_1 = a_0 \quad a_2 = \tfrac{a_1}{2} \quad a_3 = \tfrac{a_2}{3} \quad \dots \quad a_n = \tfrac{a_{n-1}}{n}$$

$$a_1 = a_0$$
$$a_2 = \tfrac{a_1}{2} = \tfrac{a_0}{2}$$
$$a_3 = \tfrac{a_2}{3} = \tfrac{a_0}{3\times 2} = \tfrac{a_0}{3!}$$
$$a_4 = \tfrac{a_3}{4} = \tfrac{a_0}{4\times 3!} = \tfrac{a_0}{4!}$$
$$...$$
$$a_n = \tfrac{a_{n-1}}{n} = \tfrac{a_0}{n\times(n-1)!} = \tfrac{a_0}{n!}$$

So the solution of $y' = y$ is

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ...\right)$$

Solution to $y' = y$ with $y(0) = 1$ has the form

$$y = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ... \right)$$

Evaluating at $x = 0$ gives $1 = y(0) = a_0(1 + 0 + 0 + 0 + ...) = a_0$ so $a_0 = 1$ and

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ...$$

But we also know the solution is $y = e^x$. Thus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ...$$

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

n	$P_n(1)$
0	1.0
1	2.0
2	2.50000
3	2.666666667
4	2.708333333
5	2.716666667
6	2.718055556
7	2.718253968
8	2.718278770
9	2.718281526
10	2.718281801
11	2.718281826
12	2.718281828

Example: A Second Order Differential Equation

$$\mathbf{y'' + xy' + y = 0}$$

$$y = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + ...$$

$$y' = a_1 + 2a_2 x + ... + n a_n x^{n-1} + (n+1) a_{n+1} x^n + (n+2) a_{n+2} x^{n+1}$$

$$xy' = a_1 x + 2a_2 x^2 + .. + n a_n x^n + (n+1) a_{n+1} x^{n+1} + (n+2) a_{n+2} x^{n+2} + ...$$

$$y'' = 2a_2 + ... + n(n-1) a_n x^{n-2} + n(n+1) a_{n+1} x^{n-1} + (n+1)(n+2) a_{n+2} x^n + ..$$

Coefficient of x^n in $y + xy' + y$ is

$$a_n + n a_n + (n+1)(n+2) a_{n+2}$$

$$= a_n(1+n) + (n+1)(n+2) a_{n+2} = (n+1)\left[a_n + (n+2) a_{n+2}\right]$$

But all coefficients must be 0. Thus

Our Equation: $y'' + xy' + y = 0$ has solution

$$y = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + ...$$

where $a_{n+2} = -\dfrac{a_n}{n+2}$

Thus

$$a_2 = -\frac{a_0}{2}, a_4 = -\frac{a_2}{4} = +\frac{a_0}{2 \cdot 4}, a_6 = -\frac{a_0}{2 \cdot 4 \cdot 6}, ...$$

$$a_3 = \frac{a_1}{3}, a_5 = +\frac{a_1}{3 \cdot 5}, a_7 = -\frac{a_1}{3 \cdot 5 \cdot 7}, ...$$

We can write solution as

$$y = a_0 \left[1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 - \frac{1}{2 \cdot 4 \cdot 6}x^6 + ... + \frac{(-1)^n}{2 \cdot 4 \cdot 6 \cdot ... \cdot 2n}x^{2n} + ... \right]$$

$$+ a_1 \left[x - \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + ... + \frac{(-1)^{n+1}}{3 \cdot 5 \cdot 7 \cdot ... \cdot (2n-1)}x^{2n-1} + ... \right]$$

where $a_0 = y(0)$ and $a_1 = y'(0)$.

Our Equation: $y'' + xy' + y = 0$ has solution

$$y = a_0 \left[1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 - \frac{1}{2 \cdot 4 \cdot 6}x^6 + ... + \frac{(-1)^n}{2 \cdot 4 \cdot 6 \cdot ... \cdot 2n}x^{2n} + ... \right]$$

$$+ a_1 \left[x - \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + ... + \frac{(-1)^{n+1}}{3 \cdot 5 \cdot 7 \cdot ... \cdot (2n-1)}x^{2n-1} + ... \right]$$

$$y = a_0 f(x) + a_1 g(x)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2 \cdot 4 \cdot 6 \cdot ... \cdot 2n}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{3 \cdot 5 \cdot 7 \cdot ... \cdot (2n-1)}$$

Example: Find Power Series Solution for $y' = 5y + 13$ with $y(0) = \frac{2}{5}$.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + a_{n+1} x^{n+1} + \ldots$$

Note first: $a_0 = \frac{2}{5}$

Write equation as $y' - 5y = 13$

Coefficient of x^n in $-5y$ is $-5a_n$

Coefficient of x^n in y' is $(n+1)a_{n+1}$

Constant Term in $y' - 5y$ is $a_1 - 5a_0$ which equals 13

So $a_1 = 13 + 5a_0 = 13 + 5\left(\frac{2}{5}\right) = 15 = 3 \times 5$

So $(n+1)a_{n+1} - 5a_n = 0$ for $n \geq 1$

Thus Recurrence Relation is

$$a_{n+1} = \frac{5a_n}{n+1}, \text{ for } n \geq 1$$

n	$a_{n+1} = \frac{5a_n}{n+1}$
1	$a_2 = \frac{1}{2}(5a_1) = \frac{1}{2}(5 \times 3 \times 5) = 3 \times \frac{5^2}{2!}$
2	$a_3 = \frac{1}{3}(5a_2) = \frac{1}{3}(5 \times 3 \times \frac{5^2}{2!}) = 3 \times \frac{5^3}{3!}$
3	$a_4 = \frac{1}{4}(5a_3) = 3 \times \frac{5^4}{4!}$
4	$a_5 = \frac{1}{5}(5a_4) = 3 \times \frac{5^5}{5!}$
...	
$n-1$	$a_n = \frac{1}{n}(5a_{n-1}) = 3 \times \frac{5^n}{n!}$

Thus $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ...a_nx^n + ..$

$$= \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + ... + 3 \times \frac{(5x)^n}{n!} + ...$$

$$y = \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + ... + 3 \times \frac{(5x)^n}{n!} + ...$$

$$= \frac{2}{5} + 3 \times \left(5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ...\right)$$

$$= \frac{2}{5} + 3 - 3 + 3 \times \left(5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ...\right)$$

$$= \frac{2}{5} - 3 + 3 \times \left(1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ...\right)$$

$$\text{so } y = -\frac{13}{5} + 3e^{5x}$$

Example: Solve $y' = 5y + 13$ with $y(0) = \frac{2}{5}$ as First Order Linear Differential Equation.

$y' - 5y = 13$. Integrating Factor is e^{-5x}

$$\left(ye^{-5x}\right)' = 13e^{-5x}$$

$$ye^{-5x} = 13\frac{-1}{5}e^{-5x} + C$$

$$y = -\frac{13}{5} + Ce^{5x}$$

Using $y(0) = \dfrac{2}{5}$ yields $C = 3$

Frequently Encountered Second Order Differential Equations in Applications

$$\mathbf{P(x)y'' + Q(x)y' + R(x)y = 0}$$

Airy	$y'' - xy = 0$
Bessel	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$
Chebyshev	$(1 - x^2)y'' - xy' + \alpha^2 y = 0$
Hermite	$y'' - 2xy' + \lambda y = 0$
Laguerre	$xy'' + (1 - x)y' + \lambda y = 0$
Legendre	$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$

Equation	Areas of Application
Airy	Acoustics, Fiber Optics
Bessel	Acoustics, Electrodynamics
Chebyshev	Approximation Theory
Hermite	Quantum Mechanics
Laguerre	Approximation Theory
Legendre	Heat Flow, Electrodynamics

**For More Material on
Power Series Solutions
of Differential Equations,
Download Chapter 9
of Brennan and Boyce**