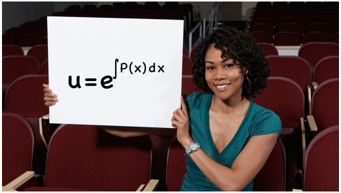
MATH 226: Differential Equations

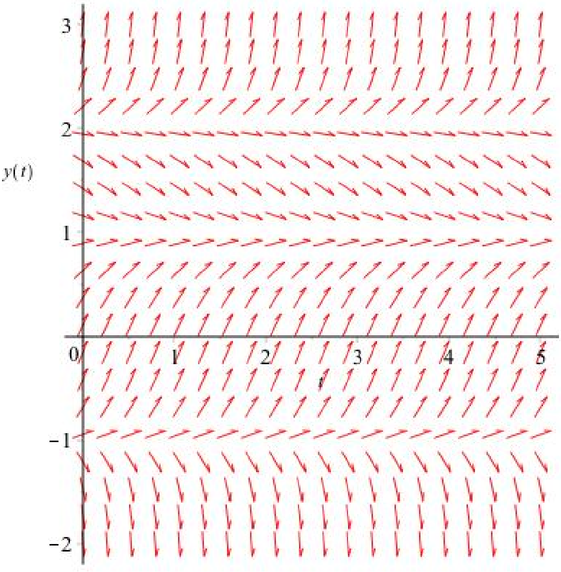


September 23, 2022

Notes on Assignment 3
Assignment 4

Prof. Kubacki Will Teach Monday's Class

Direction Field for $y' = (y-1)(y+1)(y-2)$



Another Test For Stability

Theorem: Let y^* be an equilibrium point of $y' = f(y)$ with f having a continuous derivative (as a function of y) in a neighborhood of y^*. Then

- If $f'(y^*) < 0$, then y^* is asymptotically stable
- If $f'(y^*) > 0$, then y^* is unstable
- The test is inconclusive if $f'(y*) = 0$.

Today Part I

Variables Separable

$$y'(t) = f(y)g(t)$$

- Separate Variables
- Integrate
- Use Initial Value To Find Integration Constant C
- Solve for y
- Determine Interval of Validity of Solution

Example 1: $y' = \dfrac{t^2}{y(1+t^3)}$

Note: $y \neq 0, t \neq -1$

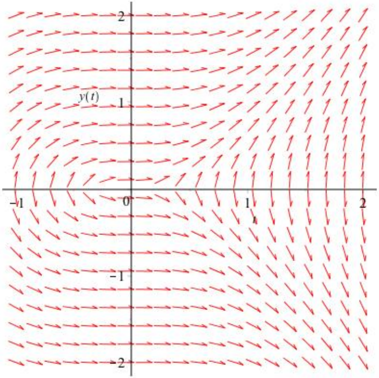
How To Solve: Separate Variables and Integrate With Respect To Independent Variable

$$y\ y' = \frac{t^2}{(1+t^3)}$$

$$\int y(t)\ y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\frac{y^2}{2} = \frac{\ln|1+t^3|}{3} + C$$

$$y^2 = \frac{2}{3}\ln|1+t^3| + C$$



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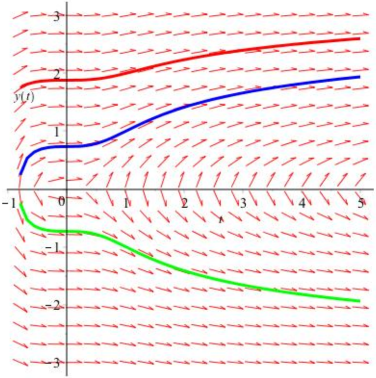


In General

$$y' = f(y)g(t)$$

Is solved as

$$\int \frac{1}{f(y)}\, y' = \int g(t)$$

$$\int \frac{1}{f(y)}\, dy = \int g(t)\, dt$$

Example: An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6$$

$$\int yy' = \int 3 - 2t$$

$$\frac{y^2}{2} = 3t - t^2 + C$$

$$y^2 = 6t - 2t^2 + C$$

$$\text{Set } t = 1, y = -6 :$$

$$36 = 6 - 2 + C \text{ so } C = 32$$

$$y^2 = -2t^2 + 6t + 32$$

Example (Continued): An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6$$

Has Solution

$$y = -\sqrt{-2t^2+6t+32}$$

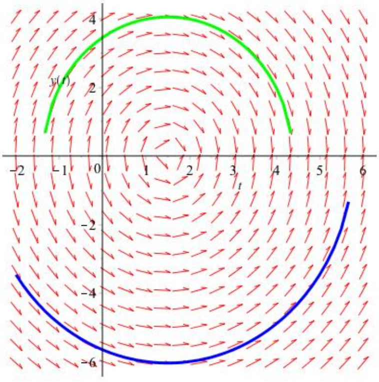
Need $-2t^2 + 6t + 32 = 2(-t^2 + 3t + 16) > 0$
or $t^2 - 3t - 16 < 0$

Roots are $t = \frac{3 \pm \sqrt{9+64}}{2} = \frac{3 \pm \sqrt{73}}{2}$
Solution is valid on $\frac{3-\sqrt{73}}{2} < t < \frac{3+\sqrt{73}}{2}$
Roughly $-2.77 < t < 5.77$.

Example (Continued): An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6 \textbf{ Blue}$$

$$y' = \frac{3-2t}{y}, y(1) = 4 \textbf{ Green}$$



Today Part II

First Order LINEAR Differential Equations

$$\frac{dy}{dt} + p(t)y = g(t)$$

Method of Integrating Factors

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First Order LINEAR Differential Equations

$$\frac{dy}{dt} + p(t)y = g(t)$$

Method of Integrating Factors

First Order LINEAR Differential Equations

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$y' + p(t)y = g(t)$$

- y' and y appear all by themselves
- No terms like y^2 or $\cos y$ or $y\, y'$
- $p(t)$ and $g(t)$ can be nonlinear, complicated, but continuous.

First Order LINEAR Differential Equations

$$y' + p(t)y = g(t)$$

Solution By Method of Integrating Factors:

Multiply Equation By Factor That Converts Left Hand Side Into a Derivative

Fundamental Theorem of Calculus

$$\text{If } f(t) = \int p(t)dt, \text{ then } f'(t) = p(t)$$

$$\text{If } f(t) = \int_0^t p(s)ds, \text{ then } f'(t) = p(t)$$

Application:

Find the derivative of $e^{\int p(t)\, dt} = exp(\int p(t)\, dt)$ with respect to t

Solution : Use Product Rule

$$\left(e^{\int p(t)\, dt}\right)' = e^{\int p(t)\, dt} p(t) = p(t) e^{\int p(t)\, dt}$$

Let's Do a Specific Example (Problem 15 in Text):

$$\text{Solve } ty' + 4y = t^2 - t + 1, \text{ with } y(1) = \frac{1}{4}$$

$$\text{Put in Standard Form (*): } y' + \frac{4}{t}y = t - 1 + \frac{1}{t}$$

$$\text{Integrating Factor is } e^{\int \frac{4}{t}\,dt} = e^{4\ln t} = e^{\ln t^4} = t^4$$

$$\text{Multiply (*) by } t^4 : t^4 y' + 4t^3 y = t^5 - t^4 + t^3$$

$$(t^4 y)' = t^5 - t^4 + t^3 \text{ and integrate}$$

$$t^4 y = \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} + C$$

$$\text{Solve for } y : y = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + Ct^{-4}$$

Original Initial Value Problem

Solve $ty' + 4y = t^2 - t + 1, \text{ with } y(1) = \frac{1}{4}$

Solution

$$y = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + \frac{C}{t^4}$$

Use initial condition to find C:

$$\frac{1}{4} = \frac{1}{6} - \frac{1}{5} + \frac{1}{4} + C$$

$$C = \frac{1}{30}$$

so

$$y = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + \frac{1}{30t^4}$$

General Case:

$$y' + p(t)y = g(t)$$

Multiply through by $e^{\int p(t)\,dt}$:

$$e^{\int p(t)\,dt}y' + p(t)e^{\int p(t)\,dt}y = g(t)e^{\int p(t)\,dt}$$

Rewrite Left Hand Side:

$$\left(e^{\int p(t)\,dt}y\right)' = g(t)e^{\int p(t)\,dt}$$

Integrate Both Sides:

$$e^{\int p(t)\,dt}y = \int g(t)e^{\int p(t)\,dt} + C$$

Divide by Coefficient of y:

$$y = e^{-\int p(t)\,dt}\int g(t)e^{\int p(t)\,dt} + Ce^{-\int p(t)\,dt}$$

$$y' + p(t)y = g(t) \text{ has solution}$$

$$y = e^{-\int p(t)\,dt} \int g(t) e^{\int p(t)\,dt} + Ce^{-\int p(t)\,dt}$$

Problem 35: Construct a first order linear differential equation whose solutions are asymptotic to the line $y = 4 - t$ as $t \to \infty$.

Solution: Add a term that goes to 0 as $t \to \infty$.

One choice would be Ce^{-t} for an arbitrary constant C.

Then solution has form $y = Ce^{-t} + 4 - t$.

Differentiate with respect to t: $y' = -Ce^{-t} - 1$

But $Ce^{-t} = y - 4 + t$

So $y' = -(y - 4 + t) - 1 = -y + 4 - t - 1 = -y + 3 - t$

$$y' + y = 3 - t$$

Next Time

Modeling With First Order Differential Equations