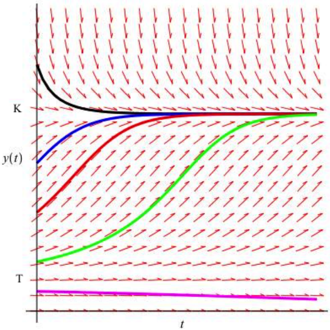
MATH 226: Differential Equations



Class 9: September 30, 2022

- Notes on Assignment 5
- Revised Assignment 6
- Two Fundamental Existence and Uniqueness Theorems (Again)

Announcements

First Team Project
Due: Next Friday

Exam 1: Wednesday, October 12

No Class Next Wednesday
Make Up Class Thursday Evening

Today's Topics

Revisit Existence and Uniqueness Theorems

for Nonlinear Differential Equations

Qualitative Analysis of Single Species Population Dynamics Autonomous Models

Existence and Uniqueness Theorems

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Notes: Theorem 2.4.1 pertains to **LINEAR** systems

Find longest open interval I containing t_o on which $p(t)$ **and** $g(t)$ are continuous.

You need not find the solution itself to find its interval of definition.

Solutions with different initial conditions do not intersect over their intervals of definition.

B.Existence and Uniqueness Theorems

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

PROOF: Follow construction of the solution

$$\phi(t) = \frac{1}{\mu(t)}[\int_{t_o}^{t} \mu(s)g(s)ds + \mu(t_o)y_o]$$

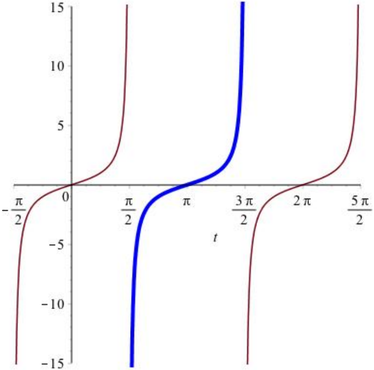
where

$$\mu(t) = e^{\int p(t)dt}$$

Example

$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

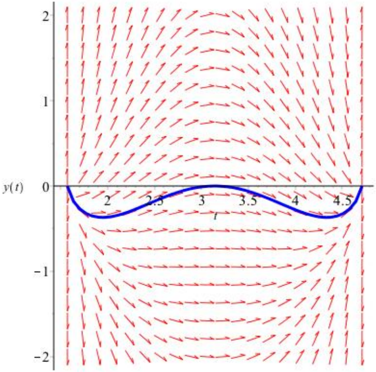
$$g(t) = \sin t$$
is continuous for all t
$$p(t) = \tan t$$
is continuous on $(\pi/2, 3\pi/2)$.



$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

Has Solution

$$y = -\cos t \ln|\cos t|$$



Theorem on Nonlinear Differential Equations

$$y' = f(t, y) \text{ with } y(t_o) = y_o$$

Suppose there is an open rectangle R in (t, y)-plane and (t_o, y_o) is in R.

IF both f and $\partial f / \partial y$ are continuous throughout R,

THEN there is some open interval I centered around t_o and a unique solution $y = \phi(t)$ of the differential equation valid on I with $\phi(t_o) = y_o$

Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

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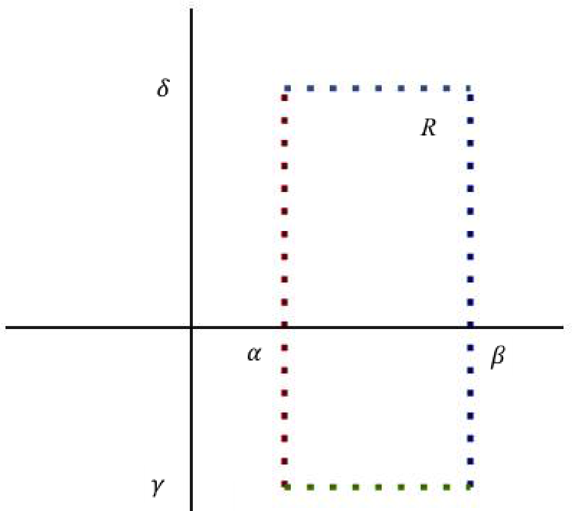

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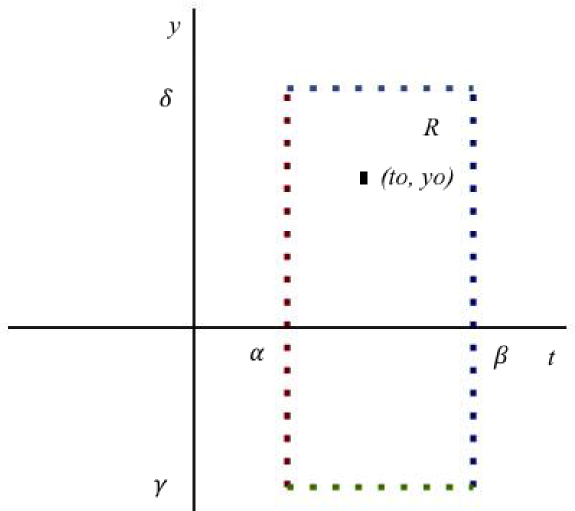

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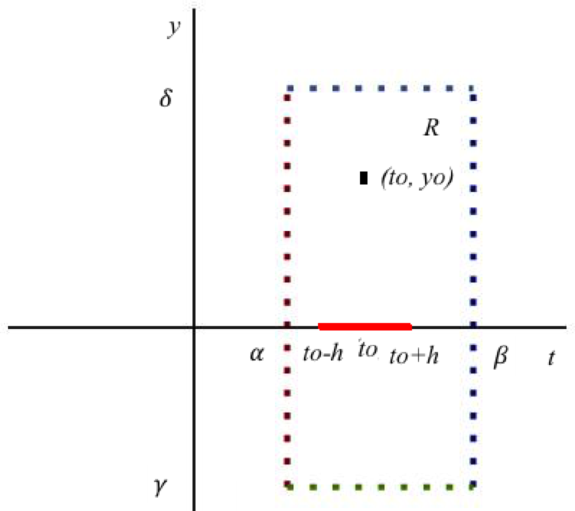


Example

$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

$$f(t, y) = \frac{\ln|ty|}{1 - t^2 + y^2}, \ \frac{\partial f}{\partial y} = \frac{(1 - t^2 + y^2)(1/y) - \ln|ty|2y}{(1 - t^2 + y^2)^2}$$

Fails to be continuous when
$t = 0$ or $y = 0$ or $1 - t^2 + y^2 = 0$.

Example

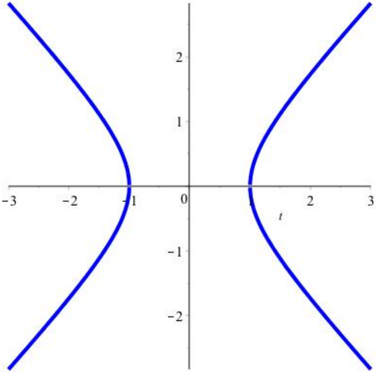
$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

Fails to be continuous when $1 - t^2 + y^2 = 0$ or $y^2 = t^2 - 1$

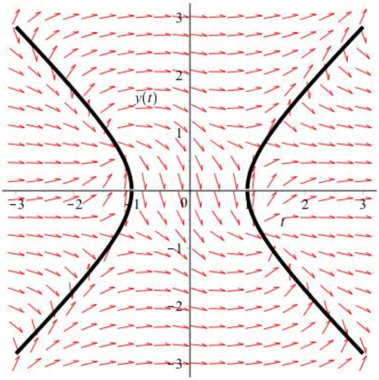
Examine $y^2 = t^2 - 1$.

Must have $t^2 - 1 \geq 0$ so $t^2 \geq 1$

So $t \geq 1$ or $t \leq -1$



$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$



Example

$$y' = y^2 \text{ with } y(0) = 3$$

$f(t, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous.

Separate Variables

$$y^{-2}y' = 1$$

$$-y^{-1} = t + C$$

Find C: $-3^{-1} = 0 + C$ so $C = -\frac{1}{3}$

$$-\frac{1}{y} = t - \frac{1}{3}$$

$$\frac{1}{y} = -t + \frac{1}{3} = \frac{-3t + 1}{3}$$

$$y = \frac{3}{1 - 3t}$$

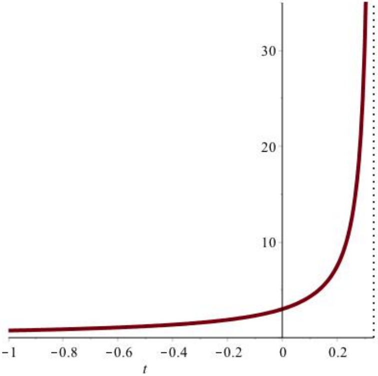
Example

$$y' = y^2 \text{ with } y(0) = 3$$

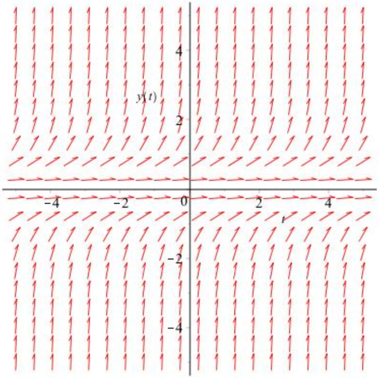
Has Solution

$$y = \frac{3}{1 - 3t}$$

Solution valid on $(-\infty, \frac{1}{3})$



$$y' = y^2$$

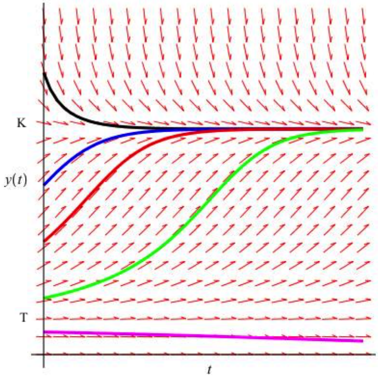


Qualitative Analysis of Single Species Population Dynamics
Autonomous Models
y' = f(y)

Logistic Growth With Threshold

$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

K is Carrying Capacity and T is Threshold.

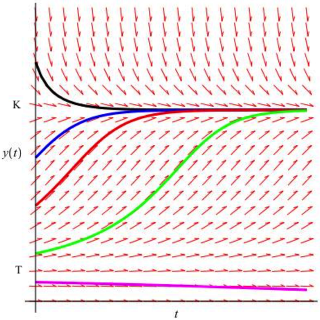


$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

$$y' = 0 \text{ when } y = 0, y = K, y = T$$

$$\begin{array}{ll} y' < 0: & 0 < y < T \\ y' > 0 & T < y < K \\ y' < 0: & y > K \end{array}$$

$$y'' = [-\frac{3}{KT}y^2 + (\frac{2}{T} + \frac{2}{K})y - 1]y'$$



Preview of Coming Attractions:

Systems of First Order Differential Equations

$$\frac{dx}{dt} = ax(t) + by(t) + f(t)$$

$$\frac{dy}{dt} = cx(t) + dy(t) + g(t)$$

Review Linear Algebra