MATH 226: Differential Equations

September 28, 2022

- Team Assignments: Project One
- Assignment 6
- Two Fundamental Existence and Uniqueness Theorems

Announcements

First Team Project

Due: Week From Friday

Meet With Me

Where to Find the Maple Files

Name	Date Modified	Size	Kind
▶ MATH0223A	Oct 11, 2021 at 3:01 PM	--	Folder
▶ MATH0223B	Oct 11, 2021 at 3:01 PM	--	Folder
▼ MATH0226A	Oct 11, 2021 at 3:01 PM	--	Folder
▶ DROPBOX	Oct 11, 2021 at 3:01 PM	--	Folder
▼ HANDOUTS	Today at 5:55 PM	--	Folder
Assignment4.maple	Today at 3:52 PM	705 KB	Maple...ocument
Class 4 Examples.mw	Feb 20, 2020 at 10:10 AM	556 KB	Maple...ocument
Class 6 Examples.mw	Today at 3:49 PM	557 KB	Maple...ocument
Direction Field for y'=(y-1)(y+1)(y-2).mw	Today at 5:51 PM	93 KB	Maple...ocument
Least Squares in Maple.mw	Today at 5:51 PM	85 KB	Maple...ocument
▶ PUBLIC_HTML	Today at 12:32 AM	--	Folder
▶ RETURN	Feb 9, 2022 at 6:58 AM	--	Folder
▶ SHARE	Oct 11, 2021 at 3:01 PM	--	Folder
▶ WORKSPACE	Feb 9, 2022 at 6:58 AM	--	Folder
▶ MATH0500J	Sep 1, 2021 at 8:38 AM	--	Folder

13. A recent college graduate borrows \$100,000 at an interest rate of 9% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800(1 + t/120)$, where t is the number of months since the loan was made.

(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

Let $S(t)$ be balance due (in dollars) on the loan at time t.
What units should we use for time?
Months or Years?
Note: Interest Rate of 9% is an **annual** rate.
Monthly rate is $\frac{.09}{12}$.
We'll use months as time unit.
Now $S' =$ rate in - rate out
Rate In: What Increases Loan Balance?
Rate In: $\frac{.09}{12}S$
Rate Out: What Decreases Loan Balance?
Rate Out: $800(1 + \frac{t}{120})$

$$S' = \frac{.09}{12}S - 800(1 + \frac{t}{120})$$

$$S' = \frac{.09}{12}S - 800(1 + \frac{t}{120})$$

First Order Linear

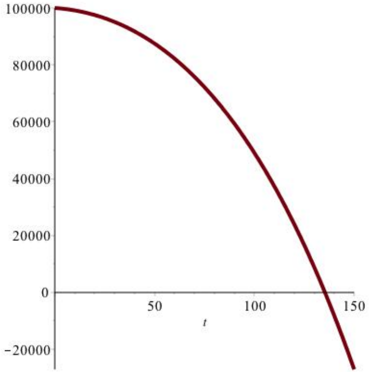
Solution is $S(t) = \frac{8000}{9}t + \frac{6080000}{27} + Ce^{\frac{3t}{400}}$

With $S(0) = 100000$, $C = -\frac{3380000}{27}$

$$S(t) = \frac{8000}{9}t + \frac{6080000}{27} - \frac{3380000}{27}e^{\frac{3t}{400}}$$

$$S(t) = 888.889t + 225{,}185.1852 - 125{,}185.1852e^{\frac{3t}{400}}$$

$$S(t) = 888.889t + 225,185.1852 - 125,185.1852e^{\frac{3t}{400}}$$



The loan will be paid when $S(t) = 0$ which happens at $t = 135.36$

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly in 20 years?

$$S(t) = 888.889t + 225,185.1852 + Ce^{\frac{3t}{400}}$$

We want to find C so that $S(20 \times 12) = S(240) = 0$
The solution is $C = -72486.62$ so

$$S(t) = 888.889t + 225,185.1852 - 72486.62e^{\frac{3t}{400}}$$

We want $S(0)$ which is 152,698.56.

25. A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5,000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in feet per second.

(a) Find the speed of the skydiver when the parachute opens.
(b) Find the distance fallen before the parachute opens.
(c) What is the limiting velocity v_L after the parachute opens?
(d) Determine how long the skydiver is in the air after the parachute opens.
(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

Distances will be measured in **feet**, time in **seconds**
With $g = 32$ ft/sec/sec, $mg = 180$, so $m = \frac{180}{32} = \frac{45}{8}$ and $\frac{1}{m} = \frac{8}{45}$.

(a) Measure the positive direction downward.
In First 10 seconds, air resistance force is).75v
Newton's Law of Motion: (mass)(acceleration) = sum of forces

$$m\frac{dv}{dt} = -0.75v + mg$$

$$\frac{dv}{dt} = -0.75\frac{v}{m} + g = -\frac{3}{4}\frac{v}{m} + 32 = -\frac{3}{4} \times \frac{8}{45}v + 32$$

$$\frac{dv}{dt} = -\frac{2}{15}v + 32$$

$$\text{Thus } v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Part (a) Continued

$$v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Solution is $v(t) = 240(1 - e^{-\frac{2}{15}t})$

At 10 seconds:

$$v(10) = 240(1 - e^{-\frac{2}{15}(10)}) = 240(1 - e^{-\frac{4}{3}}) \sim 176.7 ft/sec$$

This is the speed of the skydiver when the parachute opens

(b) Find the distance fallen before the parachute opens.
We want $s(10)$ where $s(t)$ is the number of feet fallen after t seconds.
Note: $s(0) = 0$ and $v(t) = s'(t)$ so $s(t) = \int v'(t)dt$
$s(t) = \int 240 - 240e^{-\frac{2}{15}t} = 240t + 1800e^{-\frac{2}{15}t} + C$
Since $s(0) = 0$, we have $C = -1800$.

$$\text{Thus } s(t) = 240t + 1800e^{-\frac{2}{15}t} - 1800$$

Then $s(10) = 600 + 1800e^{-\frac{4}{3}} \sim 1074.47$ feet

(c) What is the limiting velocity v_L **after** the parachute opens?
"Force of air resistance is $12|v|$ when the parachute is open."

$$m\frac{dv}{dt} = -12v + mg \text{ or } v' = -\frac{12}{m}v + 32 = -\frac{32}{15}v + 32$$

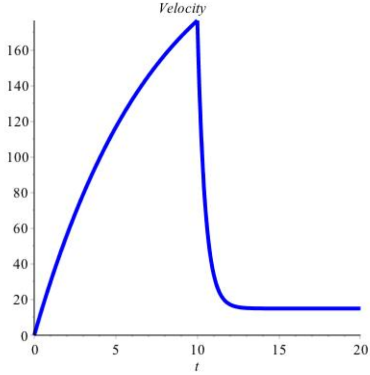
Solution of differential equation is $v(t) = 15 + Ce^{-\frac{32}{15}t}$
Limiting velocity will be 15 ft/sec regardless of the value of C.
If we start the timer again at $t = 0$ when the parachute is open, then $v(0) = 176.7$
This yields a value for C of $C = 161.7$.
Hence $v(t) = 15 + 161.7e^{-\frac{32}{15}t}$

(d) Determine how long the skydiver is in the air after the parachute opens.
Since we know the velocity, we can find the distance $s(t)$ the skydiver has fallen by integrating $v(t)$.

$$s(t) = \int v(t)dt = \int 15 + 161.7e^{-\frac{32}{15}t}dt = 15t - 75.8e^{-\frac{32}{15}t} + D$$

Now use $s(0) = 1074.5$ to find $D = 1150.3$.
The skydiver will hit the ground at time T where $s(T) = 5000$.
The answer is $T = 256.6$ seconds.

(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.



Here are *Maple* commands that would generate this graph:

$f := t \to 240 - 240 * exp(-(2/15) * t)$

$g := t \to 15 + 161.7 * exp(-(32/15) * t)$

$plot(piecewise(t < 10, f(t), g(t - 10)), t = 0..20)$

Today's Topics

Existence and Uniqueness Theorems for Linear Differential Equations

Qualitative Analysis of Single Species Population Dynamics Autonomous Models

Existence and Uniqueness Theorems

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Notes: Theorem 2.4.1 pertains to **LINEAR** systems

Find longest open interval I containing t_o on which $p(t)$ **and** $g(t)$ are continuous.

You need not find the solution itself to find its interval of definition.

Solutions with different initial conditions do not intersect over their intervals of definition.

B.Existence and Uniqueness Theorems

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

PROOF: Follow construction of the solution

$$\phi(t) = \frac{1}{\mu(t)}[\int_{t_o}^{t} \mu(s)g(s)ds + \mu(t_o)y_o]$$

where

$$\mu(t) = e^{\int p(t)dt}$$

Example

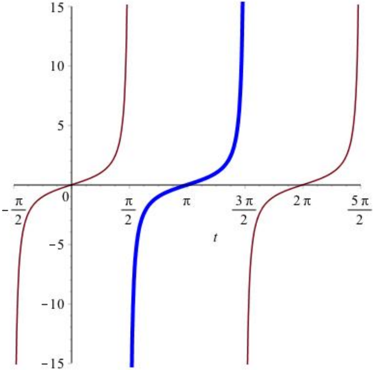
$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

$$g(t) = \sin t$$

is continuous for all t

$$p(t) = \tan t$$

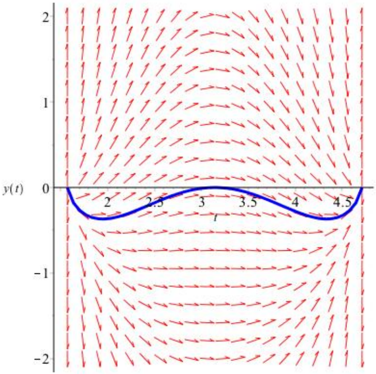
is continuous on $(\pi/2, 3\pi/2)$.



$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

Has Solution

$$y = -\cos t \ln|\cos t|$$



Theorem on Nonlinear Differential Equations

$$y' = f(t, y) \text{ with } y(t_o) = y_o$$

Suppose there is an open rectangle R in (t, y)-plane and (t_o, y_o) is in R.

IF both f and $\partial f/\partial y$ are continuous throughout R,

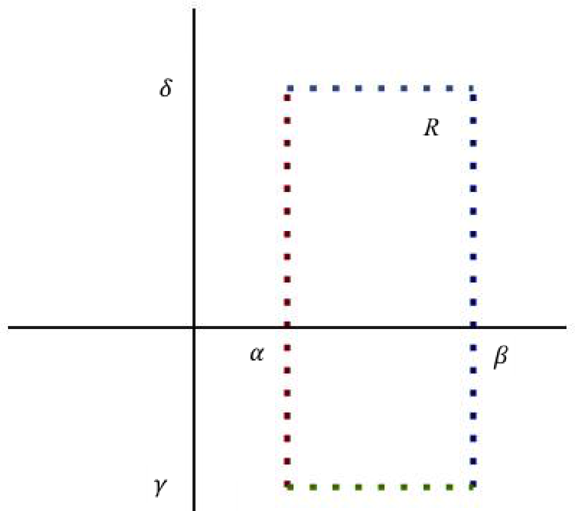
THEN there is some open interval I centered around t_o and a unique solution $y = \phi(t)$ of the differential equation valid on I with $\phi(t_o) = y_o$

Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

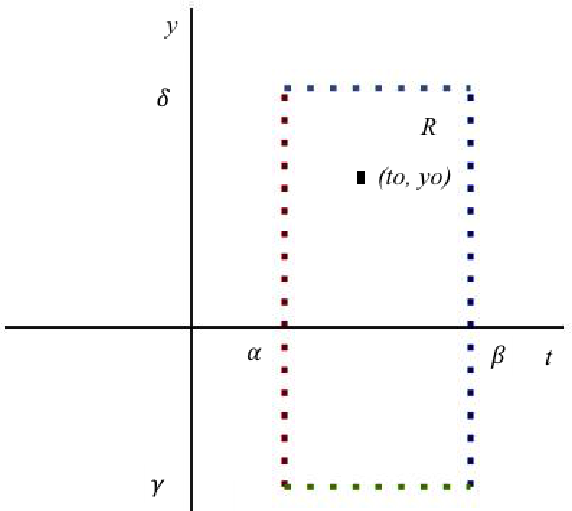
Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$



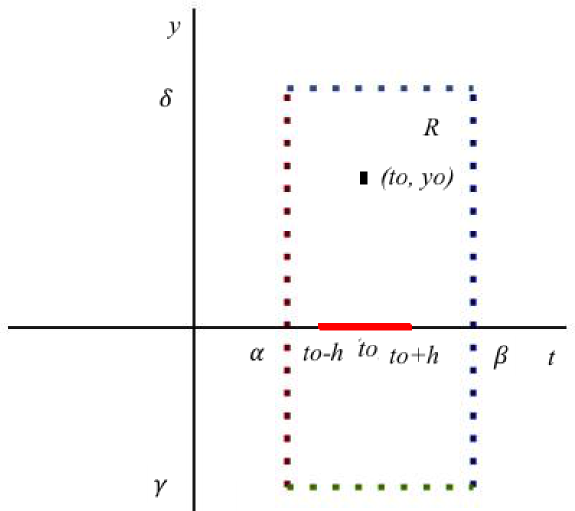
Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$



Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$



Example

$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

$$f(t,y) = \frac{\ln|ty|}{1 - t^2 + y^2}, \quad \frac{\partial f}{\partial y} = \frac{(1 - t^2 + y^2)(1/y) - \ln|ty| 2y}{(1 - t^2 + y^2)^2}$$

Fails to be continuous when
$t = 0$ or $y = 0$ or $1 - t^2 + y^2 = 0$.

Example

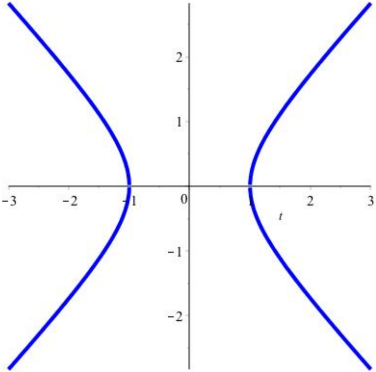
$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

Fails to be continuous when $1 - t^2 + y^2 = 0$ or $y^2 = t^2 - 1$

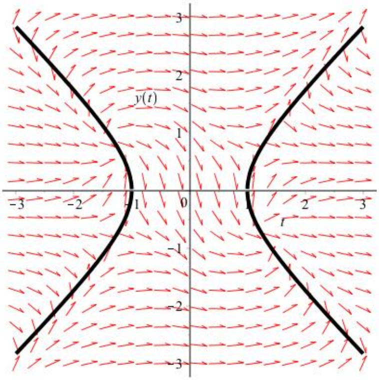
Examine $y^2 = t^2 - 1$.

Must have $t^2 - 1 \geq 0$ so $t^2 \geq 1$

So $t \geq 1$ or $t \leq -1$



$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$



Example

$$y' = y^2 \text{ with } y(0) = 3$$

$f(t, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous.

Separate Variables

$$y^{-2}y' = 1$$

$$-y^{-1} = t + C$$

Find C: $-3^{-1} = 0 + C$ so $C = -\frac{1}{3}$

$$-\frac{1}{y} = t - \frac{1}{3}$$

$$\frac{1}{y} = -t + \ \frac{1}{3} = \frac{-3t + 1}{3}$$

$$y = \frac{3}{1 - 3t}$$

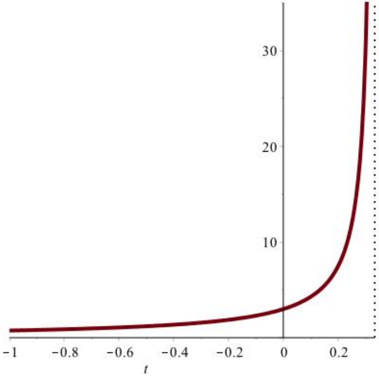
Example

$$y' = y^2 \text{ with } y(0) = 3$$

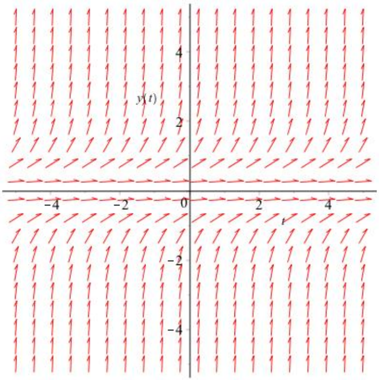
Has Solution

$$y = \frac{3}{1 - 3t}$$

Solution valid on $(-\infty, \frac{1}{3})$



$$y' = y^2$$

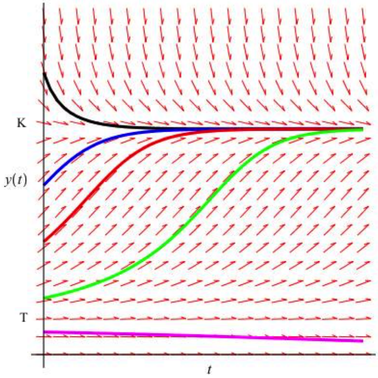


C. Qualitative Analysis of Single Species Population Dynamics Autonomous Models y' = f(y)

Logistic Growth With Threshold

$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

K is Carrying Capacity and T is Threshold.

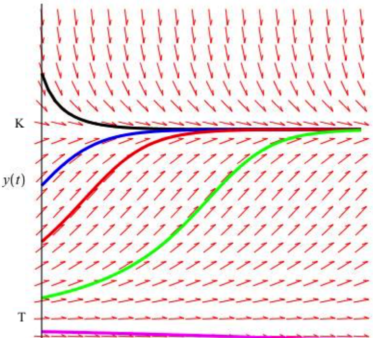


$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

$$y' = 0 \text{ when } y = 0, y = K, y = T$$

$$\begin{array}{ll} y' < 0: & 0 < y < T \\ y' > 0 & T < y < K \\ y' < 0: & y > K \end{array}$$

$$y'' = [-\frac{3}{KT}y^2 + (\frac{2}{T} + \frac{2}{K})y - 1]y'$$



Next Time:

Systems of First Order Differential Equations

$$\frac{dx}{dt} = ax(t) + by(t) + f(t)$$

$$\frac{dy}{dt} = cx(t) + dy(t) + g(t)$$

Review Linear Algebra