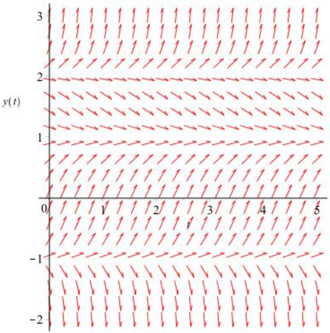
MATH 226: Differential Equations



September 21, 2022

Introductory MATLAB Worksheet Notes

Qualitative Analysis of Autonomous Differential Equation

$$\frac{dy}{dt} = f(y)$$

1. **Find Equilibrium Solutions ($f(y) = 0$)**
2. **Create Phase Line**
   Determine when $f(y) > 0$ and where $f(y) < 0$
   Label with arrows.
3. **Classify Equilibrium Solutions**
   Asymptotically Stable: $\rightarrow \bullet \leftarrow$
   Semistable: $\rightarrow \bullet \rightarrow$ or $\leftarrow \cdot \leftarrow$
   Unstable: $\leftarrow \bullet \rightarrow$

Asymptotically Stable	Semistable	Unstable
$\downarrow$	$\downarrow\uparrow$	$\uparrow$
$\bullet$	$\bullet\ \bullet$	$\bullet$
$\uparrow$	$\downarrow\ \uparrow$	$\downarrow$

4. **Sketch Solutions**
   Increasing, Decreasing, Concavity

Determining Concavity of y as a Function of t

Example From Last Time: $y' = (y-1)(y+1)(y-2) = (y^2-1)(y-2)$

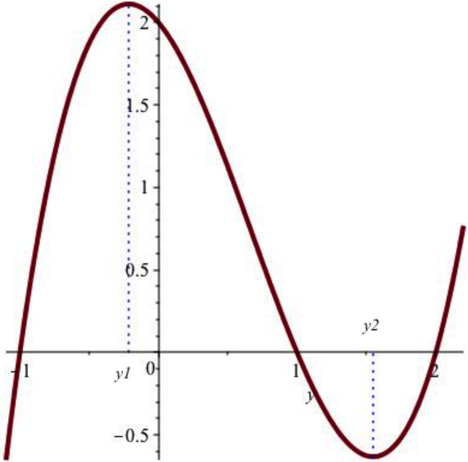
- Use Second Derivative:

$$y''(t) = f'(y(t) \times y'(t) = \frac{df}{dy} \times \frac{dy}{dt} = \frac{df}{dy} \times f(y)$$

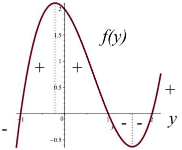
$$y'' = [3y^2 - 4y - 1][(y^2 - 1)(y2)]$$

- **Use Graph of $f(y)$ as a function of y**

Graph of $f(y) = (y^2 - 1)(y - 2)$ as a function of y



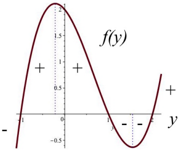
$y1 \sim -.215, y_2 \sim 1.549$



Since $y'(t) = f(y(t))$, we have $y''(t) = \frac{df}{dy}\frac{dy}{dt} = f''(y(t))y'(t)$

Interval of y values	Behavior of Derivative as function of y	$y''(t) = \frac{dy'}{dt} = \frac{dy'}{dy}\frac{dy}{dt}$
$y < -1$	Negative and Increasing	$(+)(\text{-}) = -$
$-1 < y < r_1$	Positive and Increasing	$(+)(+) = +$
$r_1 < y < 1$	Positive and Decreasing	$(\text{-})(+) = -$
$1 < y < y_2$	Negative and Decreasing	$(\text{-})(\text{-}) = +$
$y_2 < y < 2$	Negative and Increasing	$(+)(\text{-}) = -$
$y > 2$	Positive and Increasing	$(+)(+) = +$

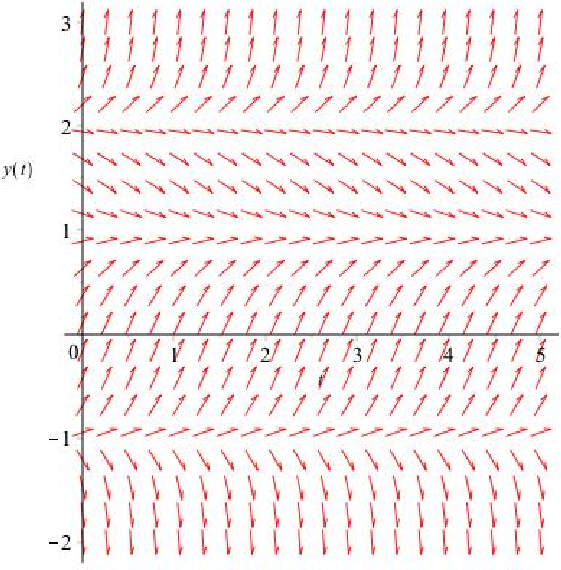
$$y_1 = \frac{2-\sqrt{7}}{3}, y_2 = \frac{2+\sqrt{7}}{3}$$



Since $y'(t) = f(y(t))$, we have $y''(t) = \frac{df}{dy}\frac{dy}{dt} = f''(y(t))y'(t)$

Interval	$y'(t)$	$y''(t)$	Behavior of Solution
$y < -1$	-	-	Decreasing Concave Down
$-1 < y < r_1$	$+$	$+$	Increasing, Concave Up
$r_1 < y < 1$	$+$	$-$	Increasing, Concave Down
$1 < y < y_2$	$-$	$+$	Decreasing, Concave Up
$y_2 < y < 2$	$-$	$-$	Decreasing, Concave Down
$y > 2$	$+$	$+$	Increasing, Concave Up

Direction Field for $y' = (y-1)(y+1)(y-2)$



Direction Field For $y' = (y-1)(y+1)(y-2)$

1) Define the Differential Equation

$DiffEq := y'(t) = (y(t) - 1) \cdot (y(t) + 1) \cdot (y(t) - 2)$

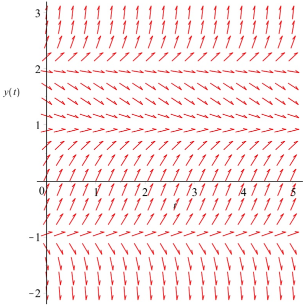
$$D(y)(t) = (y(t) - 1)\,(y(t) + 1)\,(y(t) - 2)$$

2) Read in Library of Special Tool

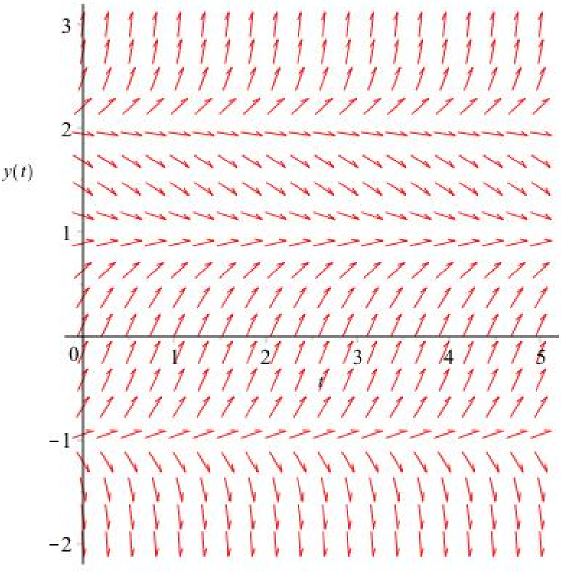
$with(DEtools):$

3) Use *DEplot* Command

$DEplot(\ DiffEq, y(t), t = 0..5, y = -2..3)$

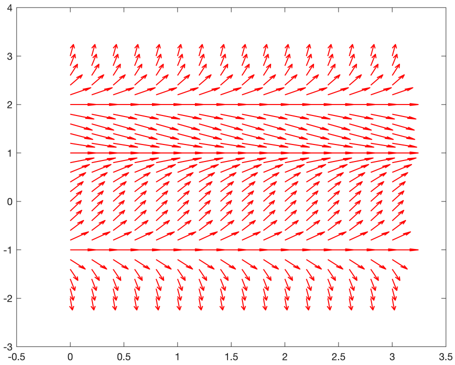


Direction Field for $y' = (y^2 - 1)(y - 2)$

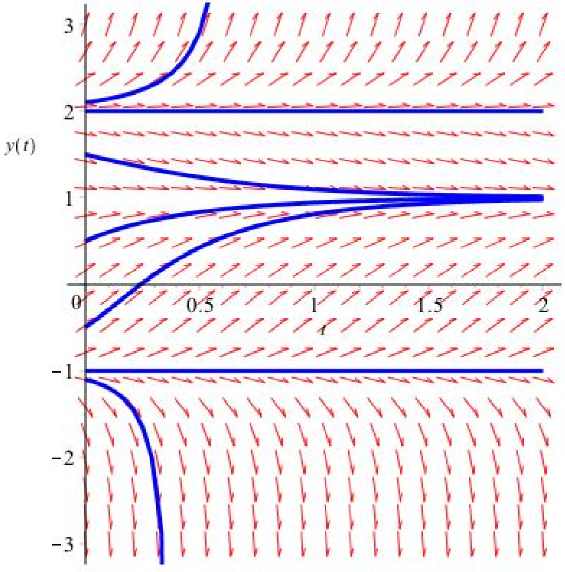


Direction Field for $y' = (y^2 - 1)(y - 2)$

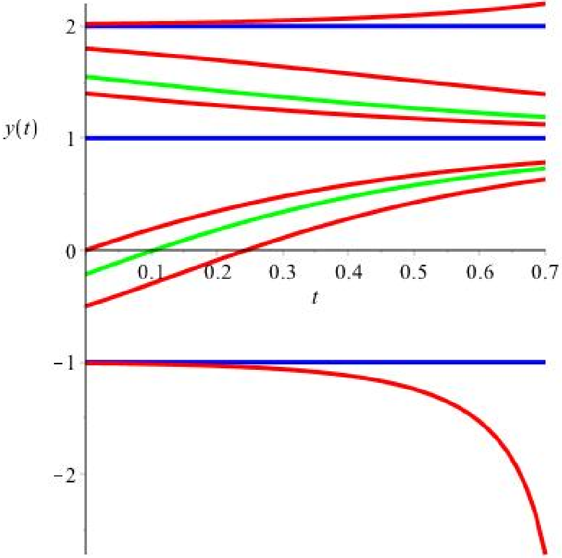
```
f =@(y) (y - 1).* (y +1).* (y-2);
[T,Y]=meshgrid(0:0.2:3, -2:0.2:3);
F = f(Y);
L = sqrt(1 + F.^2);
figure
quiver(T,Y, 1./L, F ./ L, 'r', 'LineWidth', 1);
```



Some Integral Curves For $y' = (y^2 - 1)(y - 2)$



Some More Integral Curves For $y' = (y^2 - 1)(y - 2)$



Another Test For Stability

Theorem: Let y^* be an equilibrium point of $y' = f(y)$ with f having a continuous derivative (as a function of y) in a neighborhood of y^*. Then

- If $f'(y^*) < 0$, then y^* is asymptotically stable
- If $f'(y^*) > 0$, then y^* is unstable
- The test is inconclusive if $f'(y*) = 0$.

So Far: $y' = g(t), y' = f(y)$

New: **Separable Differential Equation**

$$y' = f(y)g(t)$$

Derivative of y is product of a function of y only and a function of t only.

Example 1: $y' = \dfrac{t^2}{y(1+t^3)}$

Note: $y \neq 0, t \neq -1$
How To Solve: Separate Variables and Integrate With Respect To Independent Variable

$$y\ y' = \frac{t^2}{(1+t^3)}$$

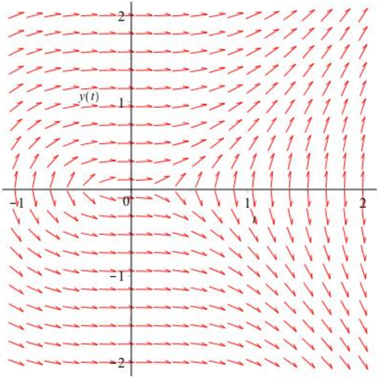
$$y(t)\ y'(t) = \frac{t^2}{(1+t^3)}$$

$$\int y(t)\ y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\int y(t)\, y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\frac{y^2}{2} = \frac{\ln|1+t^3|}{3} + C$$

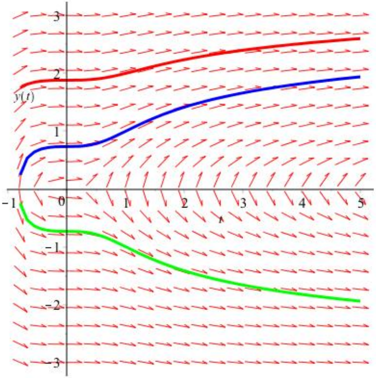
$$y^2 = \frac{2}{3}\ln|1+t^3| + C$$



$$\int y(t)\, y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\frac{y^2}{2} = \frac{\ln|1+t^3|}{3} + C$$

$$y^2 = \frac{2}{3}\ln|1+t^3| + C$$



In General

$$y' = f(y)g(t)$$

Is solved as

$$\int \frac{1}{f(y)}\, y' = \int g(t)$$

$$\int \frac{1}{f(y)}\, dy = \int g(t)\, dt$$

Example: An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6$$

$$\int yy' = \int 3 - 2t$$

$$\frac{y^2}{2} = 3t - t^2 + C$$

$$y^2 = 6t - 2t^2 + C$$

$$\text{Set } t = 1, y = -6:$$

$$36 = 6 - 2 + C \text{ so } C = 32$$

$$y^2 = -2t^2 + 6t + 32$$

Example (Continued): An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6$$

Has Solution

$$y = -\sqrt{-2t^2 + 6t + 32}$$

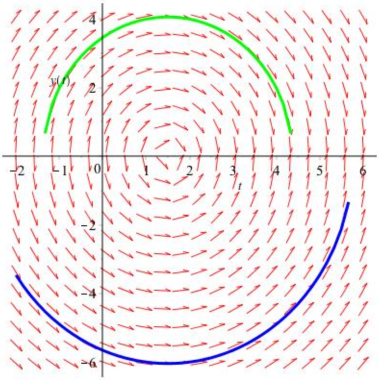
Need $-2t^2 + 6t + 32 = 2(-t^2 + 3t + 16) > 0$
or $t^2 - 3t - 16 < 0$

Roots are $t = \frac{3 \pm \sqrt{9+64}}{2} = \frac{3 \pm \sqrt{73}}{2}$
Solution is valid on $\frac{3-\sqrt{73}}{2} < t < \frac{3+\sqrt{73}}{2}$
Roughly $-2.77 < t < 5.77$.

Example (Continued): An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6 \text{ **Blue**}$$

$$y' = \frac{3-2t}{y}, y(1) = 4 \text{ **Green**}$$



Next Time