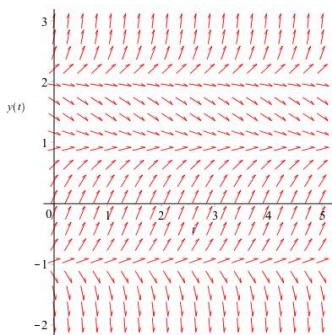


MATH 226: Differential Equations



September 19, 2022



Notes on Assignment 2

Assignment 3

Mathematician of the Day

Etta Zuber Falconer



Born: November 21, 1933

Died: September 19, 2002

MacTutor Biography

Qualitative Analysis of Autonomous Differential Equation

$$\frac{dy}{dt} = f(y)$$

1. **Find Equilibrium Solutions** ($f(y) = 0$)

2. **Create Phase Line**

Determine when $f(y) > 0$ and where $f(y) < 0$

Label with arrows.

3. **Classify Equilibrium Solutions**

Asymptotically Stable: $\rightarrow \bullet \leftarrow$

Semistable: $\rightarrow \bullet \rightarrow$ or $\leftarrow \bullet \leftarrow$

Unstable: $\leftarrow \bullet \rightarrow$

Asymptotically Stable Semistable Unstable

\downarrow

\bullet

\uparrow

$\downarrow \uparrow$

$\bullet \bullet$

$\downarrow \uparrow$

\uparrow

\bullet

\downarrow

4. **Sketch Solutions**

Increasing, Decreasing, Concavity

Determining Concavity of y as a Function of t

$$y'(t) = f(y(t)) \text{ or more simply } y' = f(y)$$

- ▶ Use Second Derivative:

$$y''(t) = f'(y(t)) \times y'(t) = \frac{df}{dy} \times \frac{dy}{dt} = \frac{df}{dy} \times f(y)$$

- ▶ Use Graph of $f(y)$ as a function of y

Determining Concavity of y as a Function of t

Example: $y' = (y - 1)(y + 1)(y - 2) = (y^2 - 1)(y - 2)$

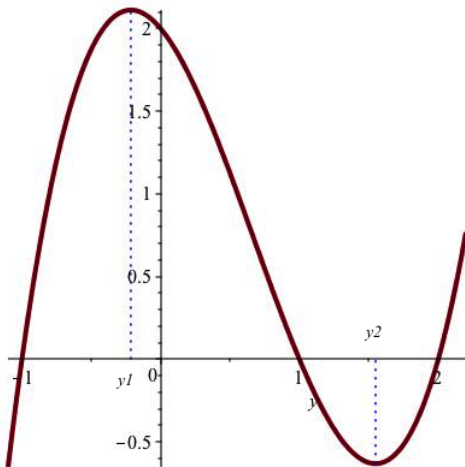
- ▶ Use Second Derivative:

$$y''(t) = f'(y(t)) \times y'(t) = \frac{df}{dy} \times \frac{dy}{dt} = \frac{df}{dy} \times f(y)$$

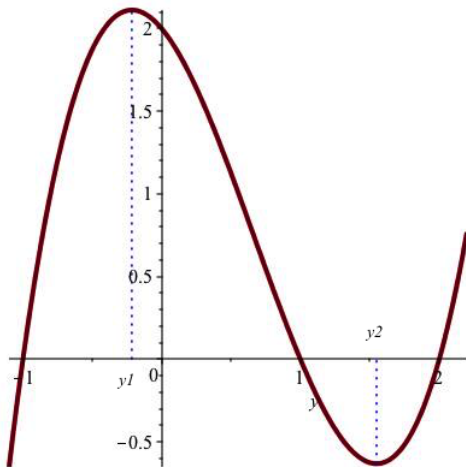
$$y'' = [3y^2 - 4y - 1][(y^2 - 1)(y - 2)]$$

- ▶ Use Graph of $f(y)$ as a function of y

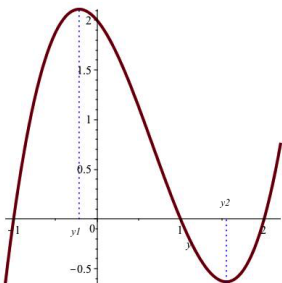
Graph of $f(y) = (y^2 - 1)(y - 2)$ as a function of y



Graph of $f(y) = (y^2 - 1)(y - 2)$ as a function of y



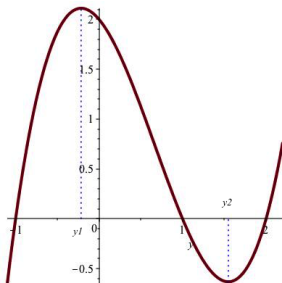
$$y_1 \sim -0.215, y_2 \sim 1.549$$



Since $y'(t) = f(y(t))$, we have $y''(t) = \frac{df}{dy} \frac{dy}{dt} = f''(y(t))y'(t)$

Interval of y values	Behavior of Derivative as function of y	$y''(t) = \frac{dy'}{dt} = \frac{dy'}{dy} \frac{dy}{dt}$
$y < -1$	Negative and Increasing	$(+)(-) = -$
$-1 < y < y_1$	Positive and Increasing	$(+)(+) = +$
$y_1 < y < 1$	Positive and Decreasing	$(-)(+) = -$
$1 < y < y_2$	Negative and Decreasing	$(-)(-) = +$
$y_2 < y < 2$	Negative and Increasing	$(+)(-) = -$
$y > 2$	Positive and Increasing	$(+)(+) = +$

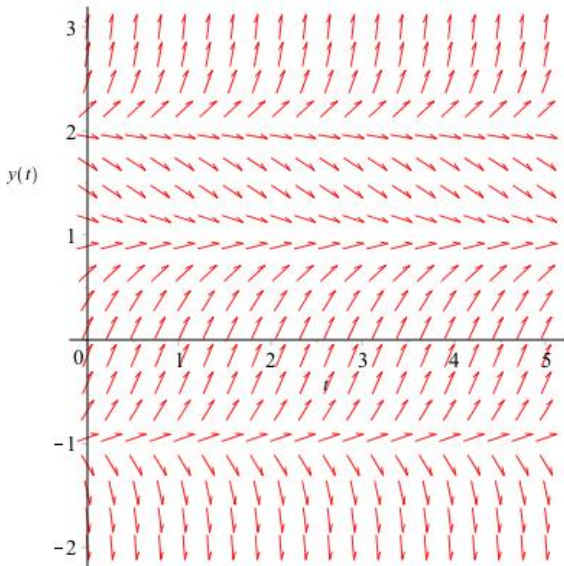
$$y_1 = \frac{2-\sqrt{7}}{3}, y_2 = \frac{2+\sqrt{7}}{3}$$



Since $y'(t) = f(y(t))$, we have $y''(t) = \frac{df}{dy} \frac{dy}{dt} = f''(y(t))y'(t)$

Interval	$y'(t)$	$y''(t)$	Behavior of Solution
$y < -1$	-	-	Decreasing Concave Down
$-1 < y < r_1$	+	+	Increasing, Concave Up
$r_1 < y < 1$	+	-	Increasing, Concave Down
$1 < y < y_2$	-	+	Decreasing, Concave Up
$y_2 < y < 2$	-	-	Decreasing, Concave Down
$y > 2$	+	+	Increasing, Concave Up

Direction Field for $y' = (y - 1)(y + 1)(y - 2)$



Direction Field For $y' = (y - 1)(y + 1)(y - 2)$

1) Define the Differential Equation

$$\text{DiffEq} := y'(t) = (y(t) - 1) \cdot (y(t) + 1) \cdot (y(t) - 2)$$

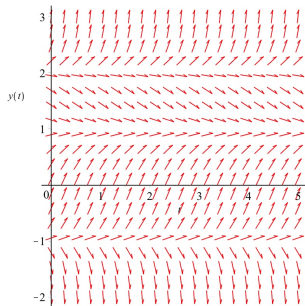
$$D(y)(t) = (y(t) - 1) (y(t) + 1) (y(t) - 2)$$

2) Read in Library of Special Tool

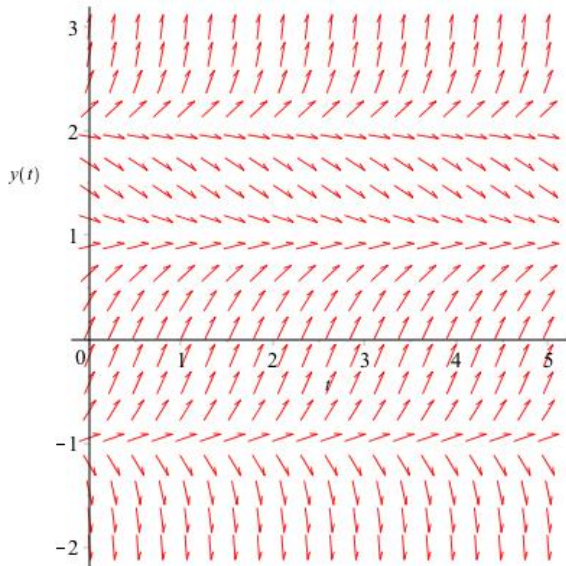
with(DEtools) :

3) Use *DEplot* Command

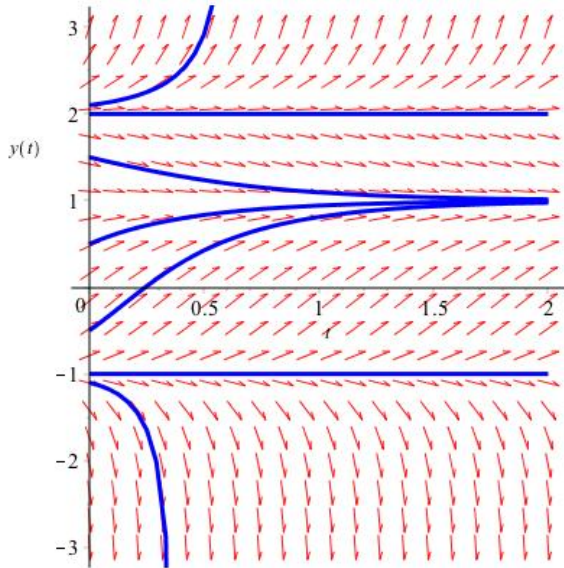
DEplot (DiffEq, y(t), t = 0 ..5, y = -2 ..3)



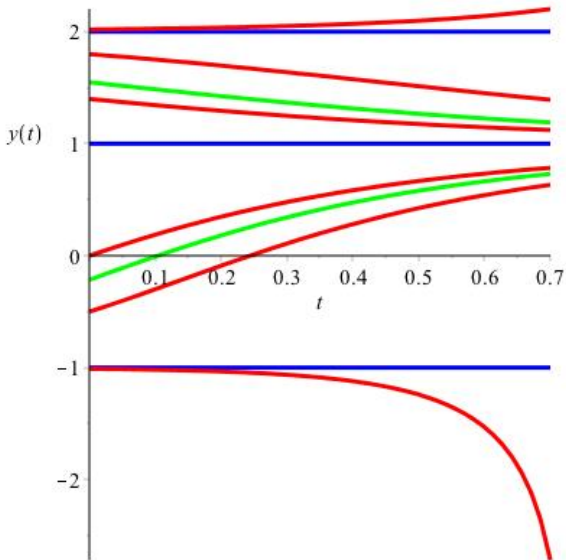
Direction Field for $y' = (y^2 - 1)(y - 2)$



Some Integral Curves For $y' = (y^2 - 1)(y - 2)$



Some More Integral Curves For $y' = (y^2 - 1)(y - 2)$



Another Test For Stability

Theorem: Let y^* be an equilibrium point of $y' = f(y)$ with f having a continuous derivative (as a function of y) in a neighborhood of y^* . Then

- ▶ If $f'(y^*) < 0$, then y^* is asymptotically stable
- ▶ If $f'(y^*) > 0$, then y^* is unstable
- ▶ The test is inconclusive if $f'(y^*) = 0$.

So Far: $y' = g(t)$, $y' = f(y)$

New: **Separable Differential Equation**

$$y' = f(y)g(t)$$

Derivative of y is product of a function of y only
and a function of t only.

$$\text{Example 1: } y' = \frac{t^2}{y(1+t^3)}$$

Note: $y \neq 0, t \neq -1$

How To Solve: Separate Variables and Integrate With Respect To Independent Variable

$$y y' = \frac{t^2}{(1+t^3)}$$

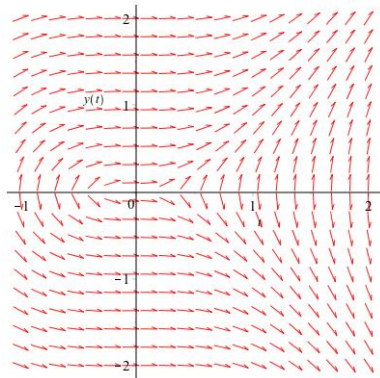
$$y(t) y'(t) = \frac{t^2}{(1+t^3)}$$

$$\int y(t) y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\int y(t) y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\frac{y^2}{2} = \frac{\ln|1+t^3|}{3} + C$$

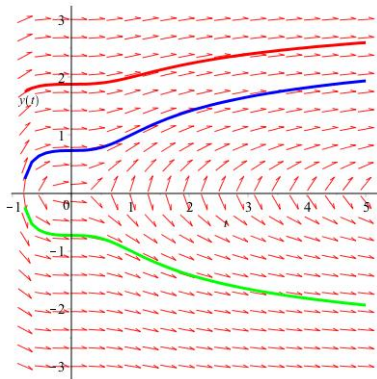
$$y^2 = \frac{2}{3} \ln|1+t^3| + C$$



$$\int y(t) y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\frac{y^2}{2} = \frac{\ln|1+t^3|}{3} + C$$

$$y^2 = \frac{2}{3} \ln|1+t^3| + C$$



In General

$$y' = f(y)g(t)$$

Is solved as

$$\int \frac{1}{f(y)} y' = \int g(t)$$

$$\int \frac{1}{f(y)} dy = \int g(t) dt$$

Example: An Initial Value Problem

$$y' = \frac{3 - 2t}{y}, y(1) = -6$$

$$\int yy' = \int 3 - 2t$$

$$\frac{y^2}{2} = 3t - t^2 + C$$

$$y^2 = 6t - 2t^2 + C$$

Set $t = 1, y = -6$:

$$36 = 6 - 2 + C \text{ so } C = 32$$

$$y^2 = -2t^2 + 6t + 32$$

Example (Continued): An Initial Value Problem

$$y' = \frac{3 - 2t}{y}, y(1) = -6$$

Has Solution

$$y = -\sqrt{-2t^2 + 6t + 32}$$

$$\begin{aligned} \text{Need } -2t^2 + 6t + 32 &= 2(-t^2 + 3t + 16) > 0 \\ \text{or } t^2 - 3t - 16 &< 0 \end{aligned}$$

$$\text{Roots are } t = \frac{3 \pm \sqrt{9+64}}{2} = \frac{3 \pm \sqrt{73}}{2}$$

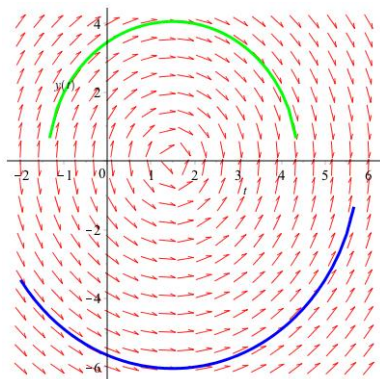
$$\text{Solution is valid on } \frac{3 - \sqrt{73}}{2} < t < \frac{3 + \sqrt{73}}{2}$$

$$\text{Roughly } -2.77 < t < 5.77.$$

Example (Continued): An Initial Value Problem

$$y' = \frac{3 - 2t}{y}, y(1) = -6 \text{ Blue}$$

$$y' = \frac{3 - 2t}{y}, y(1) = 4 \text{ Green}$$



Next Time

