MATH 226 Differential Equations

Class 32: December 2, 2022

Notes on Assignment 20
Assignment 21

Announcements
Project 3 Due
Next Friday

Euler's Method

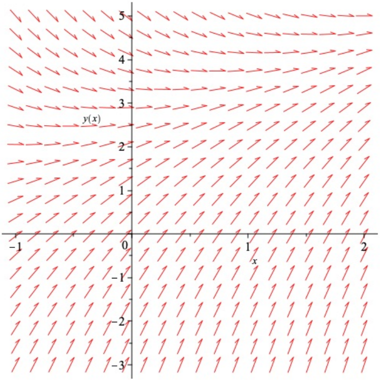
Given $y' = f(t, y)$ with $y(t_0) = y_0$

Solution: Function $\phi(t)$ so $\phi'(t) = f(t, \phi(t)), \phi(t_0) = y_0$

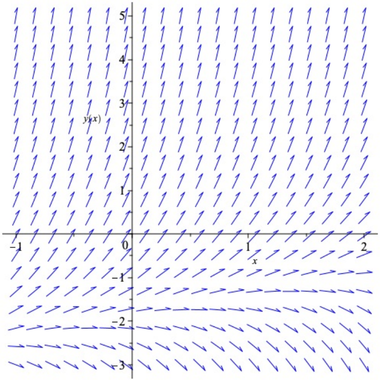
For a sequence of times $t_0, t_1, t_2, t_3, ..., t_n, t_{n+1}, ...,$
we want to estimate
$\phi(t_0), \phi(t_1), , \phi(t_2), ..., \phi(t_n), \phi(t_{n+1}), ...$
Euler: Use $y_0, y_1, y_2, ..., y_n, y_{n+1}, ...$ where

$$y_0 = \phi(t_0)$$
$$y_1 = y_0 + f(t_0, y_0)\,(t_1 - t_0) = y_0 + f(t_0, y_0)h$$
$$y_2 = y_1 + f(t_1, y_1))\,(t_2 - t_1) = y_1 + f(t_1, y_1)h$$
$$. \; . \; .$$
$$y_{n+1} = y_n + f(t_n, y_n)\,(t_{n+1} - t_n) = y_n + f(t_n, y_n)h$$

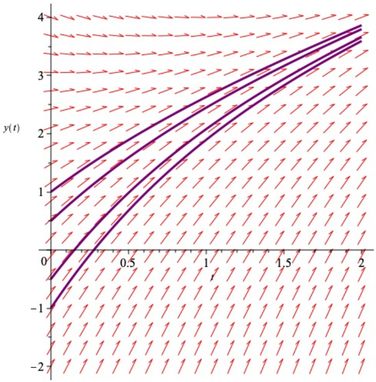
. . . (if each $t_{i+1} - t_i = h$)



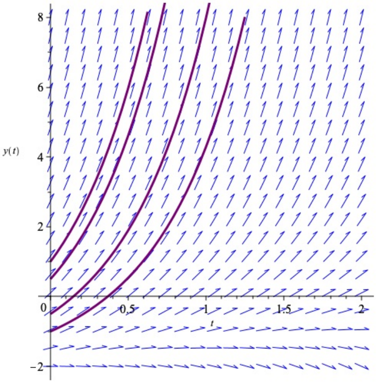
$y' = 3 + t - y$



$y' = 4 - t + 2 * t$



$y' = 3 + t - y$



$y' = 4 - t + 2 * t$

Another Way To Look at Euler's Method

The Differential Equation asserts that at each t_n, we have

$$\phi'(t_n) = f(t_n, \phi(t_n))$$

Approximate the Derivative on left by Difference Quotient:

$$\frac{\phi(t_{n+1}) - \phi(t_n)}{t_{n+1} - t_n} \approx f(t_n, \phi(t_n))$$

Using approximate values y_{n+1} and y_n, we have

$$\frac{y_{n+1} - y_n}{t_{n+1} - t_n} \approx f(t_n, \phi(t_n))$$

which is Euler's formula.

Yet Another Approach

Start with the Differential Equation

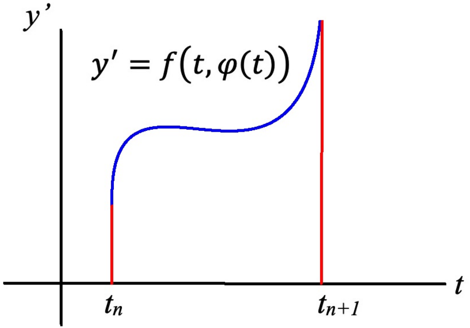
$$\phi'(t) = f(t_n, \phi(t))$$

Integrate both sides over the interval $[t_n, t_{n+1}]$:

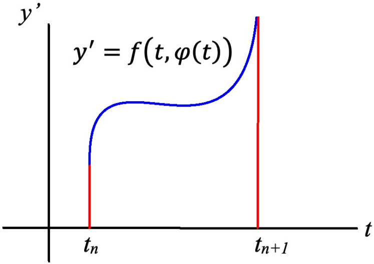
$$\int_{t_n}^{t_{n+1}} \phi'(t)\, dt = \int_{t_n}^{t_{n+1}} f(t_n, \phi(t))\, dt$$

Now left hand side is $\phi(t_{n+1}) - \phi(t_n)$

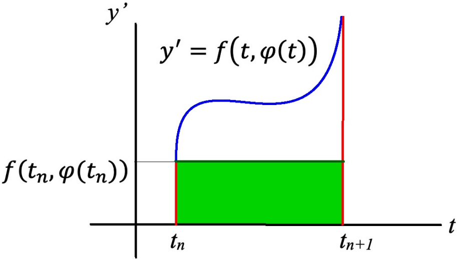
and right hand side is area under curve



Thus $\phi(t_{n+1}) - \phi(t_n) =$ Area Under Curve

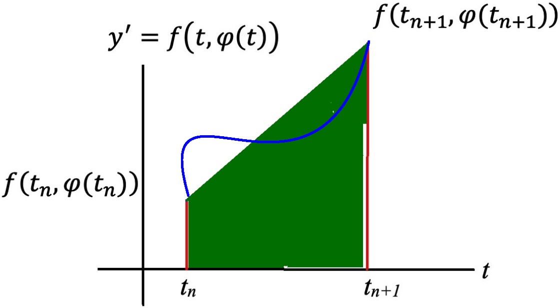


Approximate by Area of Green Rectangle $(f(t_n, \phi(t_n))(t_{n+1} - t_n)$



$$\phi(t_{n+1}) - \phi(t_n) = \text{Area Under Curve}$$

A Better Way to Estimate Area Under the Curve



$$\text{Trapezoid Rule:} A = \frac{1}{2}\left[f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1})\right](t_{n+1} - t_n)$$

$$\text{Thus } \phi(t_{n+1}) - \phi(t_n) \approx \frac{1}{2}\left[f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1})\right](t_{n+1} - t_n)$$

$$\phi(t_{n+1}) - \phi(t_n) \approx \frac{1}{2}\left[f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1})\right](t_{n+1} - t_n)$$

Replace $\phi(t)'s$ with y's:

$$y_{n+1} - y_n \approx \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right](t_{n+1} - t_n)$$

$$y_{n+1} \approx y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right](t_{n+1} - t_n)$$

But we don't know y_{n+1} so we can't find $f(t_{n+1}, y_{n+1})$

Solution: Use Euler Estimate: $y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$

Setting $h = t_{n+1} - t_n$, we have

$$y_{n+1} = y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + f(t_n, y_n)h)\right]h$$

$$y_{n+1} = y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_n + h, y_n + f(t_n, y_n)h)\right]h$$

$$y_{n+1} = y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_n + h, y_n + f(t_n, y_n)h)\right] h$$

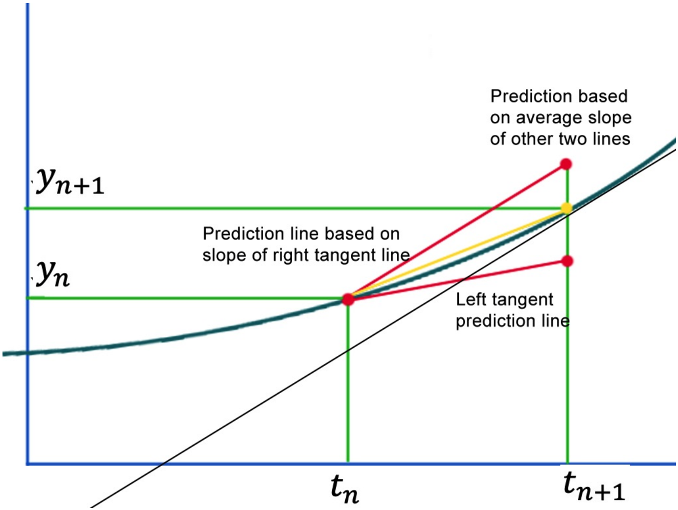
$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_n + h, y_n + f(t_n, y_n)h)}{2}\, h$$

If we let $f_n = f(t_n, y_n)$, then this formula becomes

$$y_{n+1} = y_n + \frac{f_n + f(t_n + h, y_n + f_n h)}{2}\, h$$

Improved Euler's Method or **Heun's Method**

Karl Heun
April 3, 1859 – January 10, 1929


Prediction based
on average slope
of other two lines
y_{n+1}
Prediction line based on
slope of right tangent line
y_n
Left tangent
prediction line
t_n
t_{n+1}

Heun's Method

Step 1 (Predictor): Calculate intermediate value y^*_{n+1}

$$y^*_{n+1} = y_n + hf(t_n, y_n)$$

Step 2 (Corrector): Calculate final approximation y_{n+1}

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1}\right]$$

We use Euler's Method to estimate roughly the coordinates of the next point in the solution, and with this knowledge, the original estimate is re-predicted or corrected.

Assuming that the quantity $f(t, y)$ on the right hand side of the equation can be thought of as the slope of the solution sought at any point (t, y) , we combine this with the Euler estimate of the next point to give the slope of the tangent line at the right end-point.

Next the average of both slopes is used to find the corrected coordinates of the right end interval.


Euler's method
"You guys always act like you're better than me"
Global
*Heun's method
*midpoint method
*Ralston's method
Global

Second Order Methods

$$y_{n+1} = y_n + h\left(ak_1 + bk_2\right) \text{ where}$$

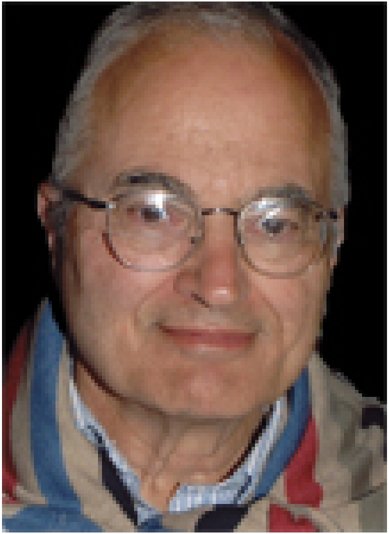
$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + ph, y_n + qk_1 h)$$

$$\text{with } a + b = 1, bp = bq = \frac{1}{2}$$

Heun	$a = b = 1/2$	$p = q = 1$
Midpoint	$a = 0, b = 1$	$p = q = 1/2$
Ralston	$a = 1/4, b = 3/4$	$p = q = 2/3$

Anthony Ralston, "Runge-Kutta Methods with Minimum Error Bounds" (1962)

Ralston Paper



Anthony Ralston
1930 –