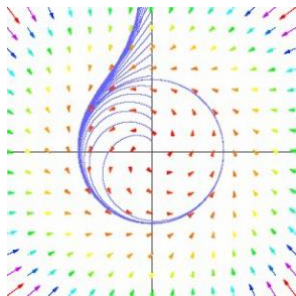


# MATH 226: Differential Equations



Class 29: November 18, 2022



Notes on Assignment 18

Assignment 19

Periodic Solutions and Limit Cycles

Converting From Cartesian To Polar

MATLAB: LimitCycle

*Maple*: LimitCycles (Handouts Folder)

*Notes on Exam 2*

## Schedule

Today:    Converting Between Cartesian and Polar  
            Coordinates  
            Periodic Solutions and Limit Cycles

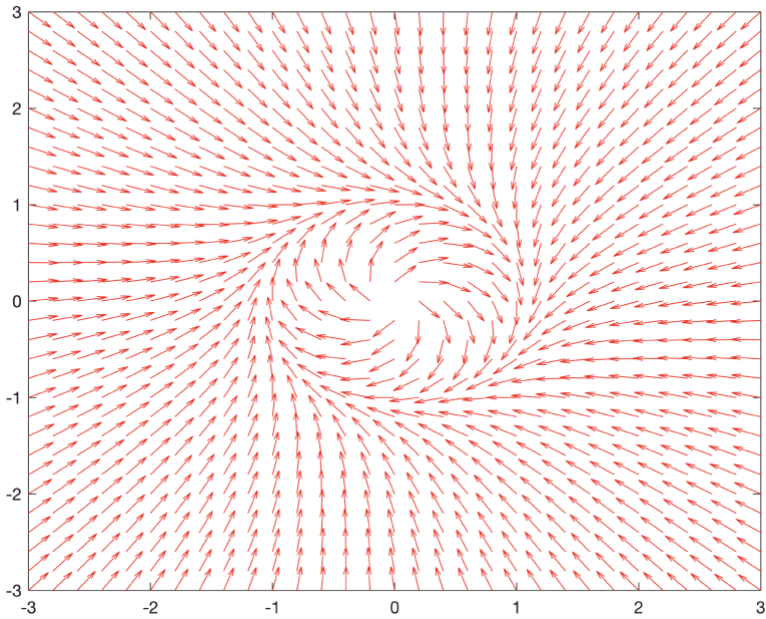
Next:    Chaos and Strange Attractors

## Limit Cycles

### A New Behavior Not Seen in Linear Systems

$$x' = y + x(1 - x^2 - y^2)$$

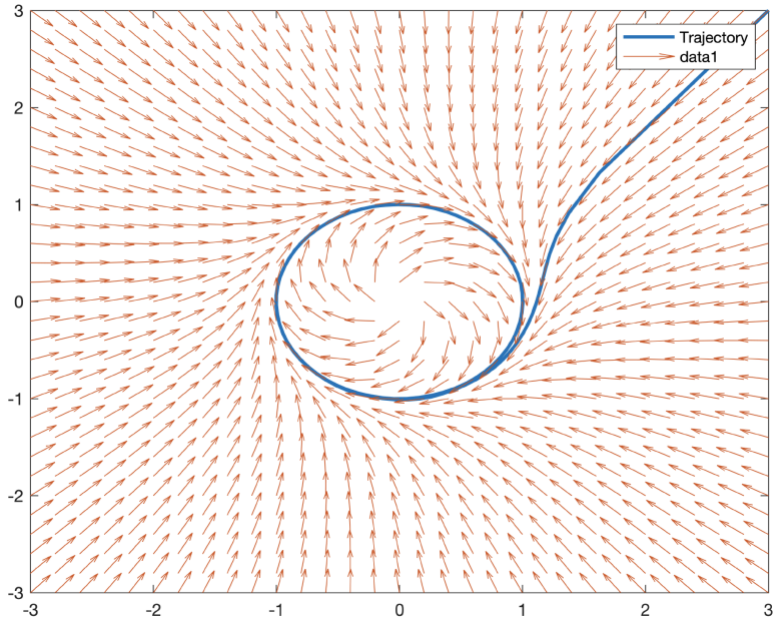
$$y' = -x + y(1 - x^2 - y^2)$$



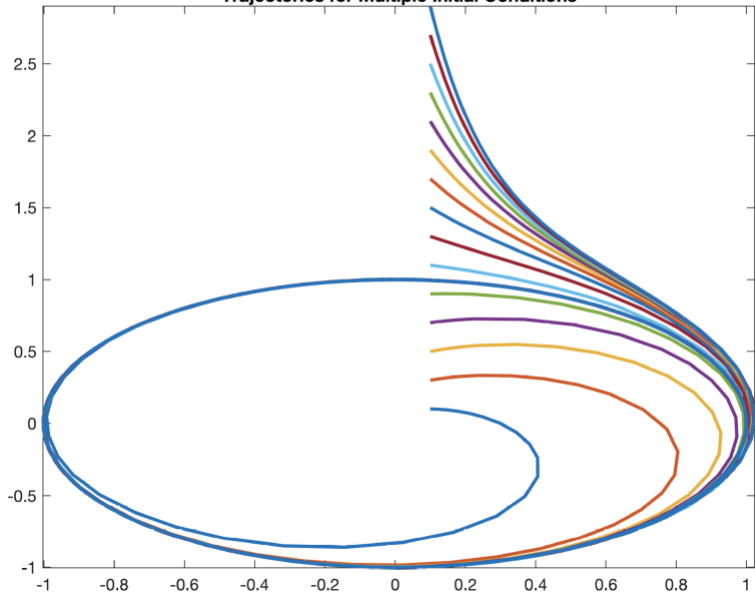
## MATLAB Code

```
[x,y]=meshgrid(-3:.2:3,-3:.2:3);  
xprime = y + x .* (1 - x.*x - y.*y);  
yprime = -x + y .* (1 - x.*x - y.*y);  
L = sqrt(xprime.^2 + yprime.^2);  
dyu=yprime./L;  
dxu=xprime./L;  
quiver(x,y,dxu,dyu, 'r')
```





Trajectories for Multiple Initial Conditions

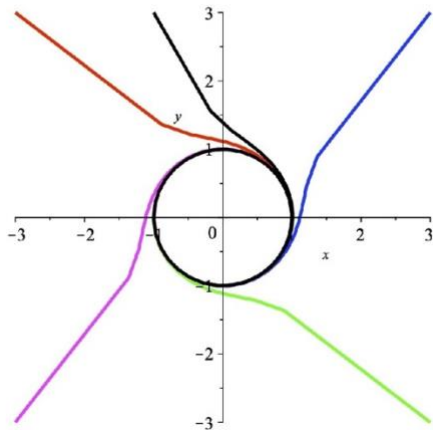




```

DEplot({ode1, ode2}, [x(t), y(t)], t=0..10, x=-3..3, y=-3..3, x(O) = -3, y(O) = -3, [x(O) = -3,
y(O) = 3], [x(O) = 3, y(O) = -3], [x(O) = -3, y(O) = -3], [x(O) = -3, y(O) = -3], [x(O) = -1, y(O) = 3], linecolor
= [blue, red, green, magenta, black], arrows = none, animate = true

```



### Polar Coordinate Version

`ode3 := r'(t) = r(t) * (1 - r(t)) :`

`ode4 := theta'(t) = -1 :`

`P1 := DEplot({ode3,ode4}, [r(t), theta(t)], t=0..10, [[theta(0) = Pi/2, r(0) = 5], [theta(0) = Pi, r(0) = 6], [theta(0) = Pi/4, r(0) = .1], [theta(0) = Pi/3, r(0) = 4]], arrows = none, linecolor = [burgundy, blue, red, green]) :`

`conv := plottools:-transform((a, b) -> [a*cos(b), a*sin(b)]) :`

`with(plots) :`

`display(conv(P1), axiscoordinates = polar) :`



## Limit Cycles

$$x' = y + x(1 - x^2 - y^2)$$

$$y' = -x + y(1 - x^2 - y^2)$$

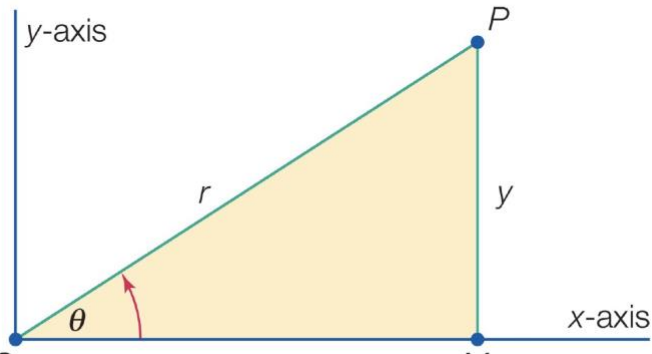
## In Polar Coordinates

$$r' = r(1 - r), \theta' = -1$$

# Converting From Cartesian To Polar

$$x = r \cos \theta$$

$$y = r \sin \theta$$



## *Converting From Cartesian To Polar*

$$x = r \cos \theta$$

$$y = r \sin \theta$$

I. Use Product and Chain Rules to Obtain

$$\frac{dx}{dt} = \frac{dr}{dt} \cos \theta + r(-\sin \theta) \frac{d\theta}{dt}$$

$$\frac{dy}{dt} = \frac{dr}{dt} \sin \theta + r(\cos \theta) \frac{d\theta}{dt}$$

$$\begin{aligned}
 \text{II.} \quad & x \frac{dx}{dt} + y \frac{dy}{dt} \\
 &= (r \cos \theta) \left[ \frac{dr}{dt} \cos \theta + r(-\sin \theta) \frac{d\theta}{dt} \right] + (r \sin \theta) \left[ \frac{dr}{dt} \sin \theta + \right. \\
 & \quad \left. r(\cos \theta) \frac{d\theta}{dt} \right] \\
 &= r \frac{dr}{dt} \cos^2 \theta - r^2 \sin \theta \cos \theta \frac{d\theta}{dt} + r \frac{dr}{dt} \sin^2 \theta \\
 & \quad + r^2 \sin \theta \cos \theta \frac{d\theta}{dt} \\
 &= r \frac{dr}{dt} \cos^2 \theta + r \frac{dr}{dt} \sin^2 \theta = r \frac{dr}{dt} [\cos^2 \theta + \sin^2 \theta] = r \frac{dr}{dt}
 \end{aligned}$$

Thus  $\boxed{r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}}$

$$\begin{aligned}
 \text{III. } & y \frac{dx}{dt} - x \frac{dy}{dt} \\
 &= \left( r \frac{dr}{dt} \sin \theta \cos \theta - r^2 \sin^2 \theta \frac{d\theta}{dt} \right) - \left( r \frac{dr}{dt} \sin \theta \cos \theta + r^2 \cos^2 \theta \frac{d\theta}{dt} \right) \\
 &= -r^2 (\sin^2 \theta + \cos^2 \theta) \frac{d\theta}{dt} \\
 &= -r^2 \frac{d\theta}{dt}
 \end{aligned}$$

Thus  $\boxed{-r^2 \frac{d\theta}{dt} = y \frac{dx}{dt} - x \frac{dy}{dt}}$

*Example:* Convert  $\begin{cases} \frac{dx}{dt} = y + x - \frac{x}{x^2+y^2} \\ \frac{dy}{dt} = -x + y - \frac{y}{x^2+y^2} \end{cases}$

*Solution:*

$$x \frac{dx}{dt} + y \frac{dy}{dt} = xy + x^2 - \frac{x^2}{x^2+y^2} + -xy + y^2 - \frac{y^2}{x^2+y^2} = x^2 + y^2 - \frac{x^2+y^2}{x^2+y^2} \\ = x^2 + y^2 - 1$$

$$\text{Hence } r \frac{dr}{dt} = r^2 - 1$$

$$\text{and } y \frac{dx}{dt} - x \frac{dy}{dt} = \left( y^2 + xy - \frac{xy}{x^2+y^2} \right) - \left( -x^2 + xy - \frac{yx}{x^2+y^2} \right) \\ = y^2 + x^2 = r^2$$

$$\text{So } -r^2 \frac{d\theta}{dt} = r^2 \text{ which yields } \frac{d\theta}{dt} = -1$$

The converted system looks like  $\begin{cases} r \frac{dr}{dt} = r^2 - 1 \\ \frac{d\theta}{dt} = -1 \end{cases}$



## Transforming Systems of Differential Equations From Cartesian To Polar Coordinates and Polar To Cartesian

$$x x' + y y' = r r'$$

$$y x' - x y' = -r^2 \theta'$$

Write As:

$$\begin{bmatrix} y & -x \\ x & y \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -r^2 \theta' \\ r r' \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y & x \\ -x & y \end{bmatrix} \begin{bmatrix} -\theta' \\ r'/r \end{bmatrix}$$

## Periodic Solutions and Limit Cycles

Definitions: A **limit cycle** is a closed trajectory in the phase plane such that other nonclosed trajectories spiral toward either from the inside or the outside (or both).

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If the trajectories on one side of the limit cycle spiral toward it while those on the other side move away as  $t \rightarrow \infty$ , then the limit cycle is **semistable**.

If the trajectories on both sides of the closed trajectory spiral away as  $t \rightarrow \infty$ , then it is called **unstable**.

**Theorem 1:** Let the functions  $F$  and  $G$  have continuous first partial derivatives in a domain  $D$  of the  $xy$ -plane.

A closed trajectory of the system

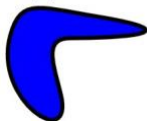
$$\begin{aligned}x' &= \frac{dx}{dt} = F(x, y), \\y' &= \frac{dy}{dt} = G(x, y)\end{aligned}$$

must necessarily enclose at least one critical point.

Moreover, if it encloses only one critical point, that point cannot be a saddle point.

Definition: A two-dimensional domain is **simply connected** if it has no holes; equivalently, any closed loop can be shrunk to a point in the domain.

Simply connected



Non-simply connected



**Theorem 2:** Let the functions  $F$  and  $G$  have continuous first partial derivatives in a simply connected domain  $D$  of the  $xy$ -plane. If  $Fx + Gy$  has the same sign throughout  $D$ , then there is no closed trajectory of the system

$$x' = \frac{dx}{dt} = F(x, y), y' = \frac{dy}{dt} = G(x, y)$$

lying entirely in  $D$ .

**Green's Theorem in the Plane:** If  $C$  is a sufficiently smooth simple closed curve that is traversed counterclockwise around a region  $R$  enclosed by  $C$ , then

$$\int_C [F(x, y) - G(x, y)] = \iint_R [F_x(x, y) + G_y(x, y)] dx dy$$

If  $F$  and  $G$  are continuous functions with continuous first partial derivatives



**Theorem 3 (Poincaré-Bendixson Theorem):** Let the functions  $F$  and  $G$  have continuous first partial derivatives in a domain  $D$  of the  $xy$ -plane.

Let  $D_1$  be a bounded subdomain in  $D$ , and let  $R$  be the region that consists of  $D_1$  plus its boundary (all points of  $R$  are in  $D$ ).

Suppose that  $R$  contains no critical points of the system

$$\int_C [F(x, y) - G(x, y)] = \iint_R [F_x(x, y) + G_y(x, y)] dx dy$$

If there exists a constant  $t_0$  such that  $x = \varphi(t)$ ,  $y = \psi(t)$  is a solution of the system that exists and stays in  $R$  for all  $t \geq t_0$ ,  
then either

$x = \varphi(t)$ ,  $y = \psi(t)$  is a periodic solution with closed trajectory or  
 $x = \varphi(t)$ ,  $y = \psi(t)$  has a trajectory that spirals toward a closed trajectory as  $t \rightarrow \infty$

In either case, the system has a periodic solution in  $R$ .

