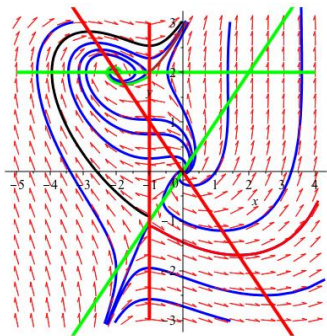


# MATH 226: Differential Equations



Class 24: November 7, 2022



Notes on Assignment 15

Assignment 16

Variation of Parameters (MATLAB) Variation of

Parameters 1 (Maple)

Variation of Parameters 2 (Maple)

## Today's Agenda

- ▶ Review Matrix Exponential
- ▶ Nonhomogenous Systems
- ▶ More on Defective Matrices
- ▶ Quick Peek at Nonlinear Systems

## Review

# The Matrix Exponential

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots + A^k \frac{t^k}{k!} + \dots$$

$e^{At}$  is an  $n \times n$  matrix

Each column of  $e^{At}$  is a solution of  $\mathbf{x}' = A\mathbf{x}$

The columns form a linearly independent set

Some Other Nice Properties:

$$e^{A \times 0} = I$$

$$(e^{At})' = Ae^{At}$$

$$e^{-At} = (e^{At})^{-1}$$

$$e^{A(r+s)} = e^{Ar} e^{As}$$

## *Review*

$e^{At}$  has wonderful properties but it is hard to compute via the power series definition.

**Alternate Way To Compute Matrix Exponential  $e^{At}$**

$$e^{At} = X(t)(X(0))^{-1}$$

**where  $X(t)$  is any Fundamental Matrix for  $x' = Ax$ .**

How To Find  $X$ ?

Use eigenvalue/eigenvector approach.

## Nonhomogeneous Systems

Recall Solution of  $x' = ax + g(t)$

$$x' - ax = g(t)$$

Multiply by integrating factor  $e^{-at}$

$$(xe^{-at})' = e^{-at}g(t)$$

$$xe^{-at} = \int e^{-at}g(t) dt + C$$

$$x = e^{at} \int e^{-at}g(t) dt + Ce^{at}$$

$$x = e^{at} \int_0^t e^{-as}g(s) ds + Ce^{at}$$

Evaluate at  $t = 0$ :

$$x = e^{at} \int_0^t e^{-as}g(s) ds + x(0)e^{at}$$

## Nonhomogeneous Systems

$x' = ax + g(t)$  has solution  $x = e^{at} \int_0^t e^{-as} g(s) ds + e^{at} x(0)$

$\mathbf{X}' = A\mathbf{X} + \mathbf{g}(t)$  has solution

$$\mathbf{X} = e^{At} \int_0^t e^{-As} \mathbf{g}(s) ds + e^{At} \mathbf{X}(0)$$

$$\mathbf{X} = \Phi(t) \int_0^t \Phi^{-1}(s) \mathbf{g}(s) ds + \Phi(t) \mathbf{X}(0)$$

We can also write the solution as

$$\mathbf{X}(t) \int_{t_0}^t \mathbf{X}^{-1}(s) \mathbf{g}(s) ds + \mathbf{X}(t) \mathbf{X}^{-1}(t_0)$$

where  $\mathbf{X}$  is any fundamental solution of  $\mathbf{X}' = A\mathbf{X}$

## More on Defective Matrices

Solving  $\mathbf{x}' = A\mathbf{x}$  when  $A$  is "defective"

Suppose  $\lambda$  is an eigenvalue of  $A$  with algebraic multiplicity 3 but geometric multiplicity 1.

Find  $\mathbf{v}$  such that  $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Find  $\mathbf{w}$  such that  $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Find  $\mathbf{u}$  such that  $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Then 3 linearly independent solutions of  $\mathbf{x}' = A\mathbf{x}$  are

$$e^{\lambda t}\mathbf{v}$$

$$te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$

$$\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$$

What to do if  $\lambda$  is an eigenvalue of  $A$  with algebraic multiplicity **bigger than 3** but geometric multiplicity 1?



## WHY DOES THIS WORK?

### KEY STEP:

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \text{ implies } A\mathbf{v} = \lambda\mathbf{v}$$

$$(A - \lambda I)\mathbf{w} = \mathbf{v} \text{ implies } A\mathbf{w} = \mathbf{v} + \lambda\mathbf{w}$$

$$(A - \lambda I)\mathbf{u} = \mathbf{w} \text{ implies } A\mathbf{u} = \mathbf{w} + \lambda\mathbf{u}$$

Find  $\mathbf{v}$  such that  $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Find  $\mathbf{w}$  such that  $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Find  $\mathbf{u}$  such that  $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Note: 3 systems of linear algebraic equations with the **same coefficient matrix**

Example 
$$\begin{pmatrix} 4 & 8 & 12 \\ -82 & 204 & 295 \\ 64 & -128 & -184 \end{pmatrix}$$

Characteristic Polynomial :  $\lambda^3 - 24\lambda^2 + 192\lambda - 512 = (\lambda - 8)^3$

Eigenvalue  $\lambda = 8$  has algebraic multiplicity 3, but geometric multiplicity only 1.

Here  $(A - \lambda I) = \begin{pmatrix} -4 & 8 & 12 \\ -82 & 196 & 295 \\ 64 & -128 & -192 \end{pmatrix}$

To solve  $(A - \lambda I)\mathbf{x} = \mathbf{b}$  construct augmented matrix

$$\left( \begin{array}{ccc|c} -4 & 8 & 12 & a \\ -82 & 196 & 295 & b \\ 64 & -128 & -192 & c \end{array} \right)$$

and reduce to row echelon form

To solve  $(A - \lambda I)\mathbf{x} = \mathbf{b}$  :

Augmented matrix is  $\left( \begin{array}{ccc|c} -4 & 8 & 12 & a \\ -82 & 196 & 295 & b \\ 64 & -128 & -192 & c \end{array} \right)$

$$\left( \begin{array}{ccc|c} 1 & 0 & \frac{1}{16} & \frac{49}{32}a + \frac{1}{16}b \\ 0 & 1 & \frac{49}{32} & -\frac{41}{64}a + \frac{1}{32}b \\ 0 & 0 & 0 & 16a + c \end{array} \right) \text{ so } \mathbf{x} = \left( \begin{array}{c} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ 16a + c \end{array} \right)$$

For  $(A - \lambda I)\mathbf{v} = \mathbf{0}$ , set  $a = 0, b = 0, c = 0$ :

$$\mathbf{v} = \begin{pmatrix} -(1/16)x_3 \\ -(49/32)x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1/16 \\ -49/32 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -49 \\ 32 \end{pmatrix} \text{ if } x_3 = 1$$

$$\left( \begin{array}{ccc|c} 1 & 0 & \frac{1}{16} & \frac{49}{32}a + \frac{1}{16}b \\ 0 & 1 & \frac{49}{32} & -\frac{41}{64}a + \frac{1}{32}b \\ 0 & 0 & 0 & 16a + c \end{array} \right) \text{ so } \mathbf{x} = \begin{pmatrix} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ 16a + c \end{pmatrix}$$

For  $(A - \lambda I)\mathbf{w} = \mathbf{v}$ , set  $a = -2, b = -49, c = 32$ :

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} (-1/16)w_3 \\ 1/4 - (49/32)w_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ 0 \end{pmatrix} \text{ if } w_3 = 0.$$

$$\mathbf{x} = \begin{pmatrix} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ 16a + c \end{pmatrix}$$

For  $(A - \lambda I)\mathbf{u} = \mathbf{w}$ , set  $a = 0$ ,  $b = -1/4$ ,  $c = 0$ :

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (-1/64) - (1/16)u_3 \\ (-1/128) - (49/32)u_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} -1/64 \\ -1/128 \\ 0 \end{pmatrix} \text{ if } u_3 = 0.$$

$$\text{Thus } \mathbf{v} = \begin{pmatrix} -2 \\ -49 \\ 32 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1/4 \\ 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -1/64 \\ -1/128 \\ 0 \end{pmatrix}$$

Solving  $\mathbf{x}' = A\mathbf{x}$  when  $A$  is "defective"

Suppose  $\lambda$  is an eigenvalue of  $A$  with algebraic multiplicity 4 but geometric multiplicity 1.

Find  $\mathbf{v}$  such that  $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Find  $\mathbf{w}$  such that  $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Find  $\mathbf{u}$  such that  $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Find  $\mathbf{z}$  such that  $(A - \lambda I)\mathbf{z} = \mathbf{u}$

Then 4 linearly independent solutions of  $\mathbf{x}' = A\mathbf{x}$  are

$$e^{\lambda t}\mathbf{v}$$

$$te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$

$$\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$$

$$\frac{t^3}{3!}e^{\lambda t}\mathbf{v} + \frac{t^2}{2}e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{u} + e^{\lambda t}\mathbf{z}$$

**Next Major Goal:**  
**Study Nonlinear Systems**  
**of General First Order Differential Equations**

$$\begin{array}{ll} x' = F(x, y, t) & x' = F(x, y, z, t) \\ y' = G(x, y, t) & y' = G(x, y, z, t) \\ & z' = H(x, y, z, t) \end{array}$$

**General Case**

$$\begin{array}{l} x_1' = f_1(x_1, x_2, \dots, x_n, t) \\ x_2' = f_2(x_1, x_2, \dots, x_n, t) \\ \cdot \\ \cdot \\ \cdot \\ x_n' = f_n(x_1, x_2, \dots, x_n, t) \end{array}$$

$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

### Autonomous Systems

No Explicit  $t$  on Right Hand Side

$n = 2$	$n = 3$
$x' = F(x, y)$	$x' = F(x, y, z)$
$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$



## Three Approaches:

**Analytic: Rarely Possible To Find Closed Form Solution**

**Numeric: Detailed Information About a Single Solution**

**Geometric: Qualitative Information About All Solutions**

## Example

$$x' = (2 - y)(x - y)$$

$$y' = (1 + x)(x + y)$$

STEP ONE: Identify All Equilibrium Points

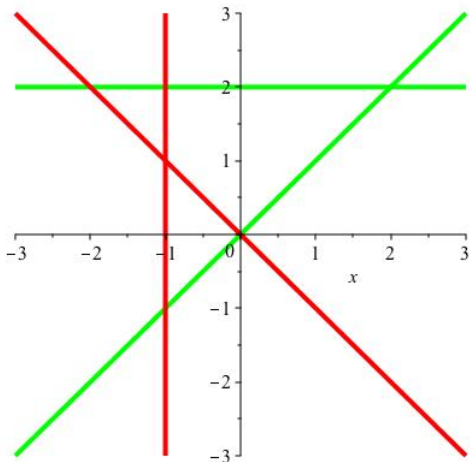
**$x' = 0$  along lines  $y = 2$  and  $y = x$**

**$y' = 0$  along lines  $x = -1$  and  $y = -x$**

$$x' = (2 - y)(x - y), y' = (1 + x)(x + y)$$

$x' = 0$  along lines  $y = 2$  and  $y = x$

$y' = 0$  along lines  $x = -1$  and  $y = -x$



$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

### Autonomous Systems

No Explicit  $t$  on Right Hand Side

$n = 2$	$n = 3$
$x' = F(x, y)$	$x' = F(x, y, z)$
$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$

## Special Properties of Autonomous Systems

1. Direction Field is Independent of Time
2. Only One Trajectory Passing Through Each Point  $(x_0, y_0)$
3. A Trajectory Can Not Cross Itself
4. A Single Well-Chosen Phase Portrait Simultaneously Displays Important Information About All Solutions

## Analyzing a Nonlinear System

Our Example From Last Time:

$$x' = F(x, y) = (2 - y)(x - y) = 2x - 2y - xy + y^2$$

$$y' = G(x, y) = (1 + x)(x + y) = x + y + x^2 + xy$$

**Step 1:** Find Critical Points:  $x' = 0$  and  $y' = 0$

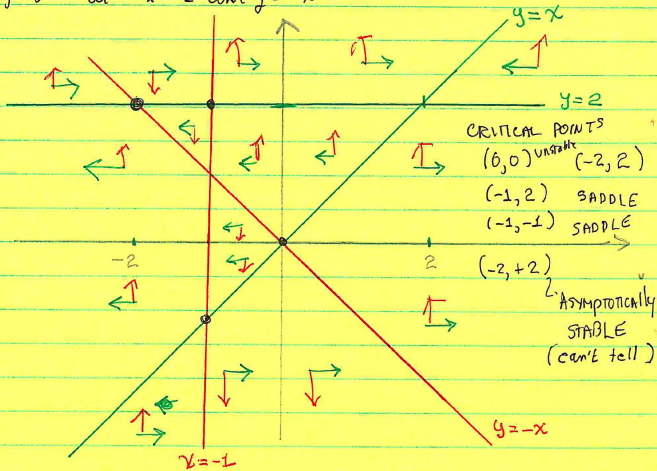
**Step 2:** Find Signs of  $x'$  and  $y'$

EXAMPLE  $x' = (2-y)(x-y) = 2x - 2y - xy + y^2$

$y' = (1+x)(x+y) = x + y + x^2 + xy$

$x' = 0$  at  $y = 2$  and  $y = x$  GREEN

$y' = 0$  at  $x = -1$  and  $y = -x$



**Current Goal:**  
**Approximating Nonlinear Autonomous  
System with Linear System  
Near An Equilibrium Point**

$$x' = F(x, y)$$

$$y' = G(x, y)$$

$$F(x, y) \approx F(x^*, y^*) + F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$

$$G(x, y) \approx G(x^*, y^*) + G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

But at Equilibrium Points,  $F(x^*, y^*) = 0$ ,  $G(x^*, y^*) = 0$  so

$$F(x, y) \approx F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$

$$G(x, y) \approx G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$





Carl Gustav Jacob Jacobi  
December 10, 1804 – February 18, 1851

$$F(x, y) \approx F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

which we can write as

$$\begin{bmatrix} F_x(x^*, y^*) & F_y(x^*, y^*) \\ G_x(x^*, y^*) & G_y(x^*, y^*) \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$$

or

$$J(x^*, y^*) \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} = J(x^*, y^*) \begin{bmatrix} h \\ k \end{bmatrix}$$

$J$  is called the **Jacobi Matrix** or **Jacobian**