MATH 226: Differential Equations

Class 20: October 28, 2022

Notes on Assignment 12

Assignment 13

Political Movement Model in *Maple* (in Handouts Folder)

Political Movement Model in MATLAB

Announcements

- Second Project Due Friday, November 4
- Exam 2 on November

Review of $\mathbf{X}' = A\mathbf{X}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Possibilities

$e^{\lambda t}\mathbf{v}, e^{\mu t}\mathbf{w}$

2 Complex Roots: $\lambda = u + iv, u - iv$

$e^{ut}(\mathbf{a}\cos vt - \mathbf{b}\sin vt), e^{ut}(\mathbf{a}\sin vt + \mathbf{b}\cos vt)$

where $\mathbf{a} + i\mathbf{b}$ is an eigenvector of λ.

1 Real Double Root

$e^{\lambda t}\mathbf{v},\ te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$ where $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Complex Eigenvalue

$$\lambda = u + iv$$

Solutions Look Like

$$e^{ut}(\mathbf{a}\cos vt - \mathbf{b}\sin vt) \text{ and } e^{ut}(\mathbf{a}\sin vt + \mathbf{b}\cos vt)$$

where $\mathbf{a} + i\mathbf{b}$ is an eigenvector

Long Term Qualitative Behavior depends on sign of u

$u > 0$	Spiral Source
$u < 0$	Spiral Sink
$u = 0$	Center

Systems of First Order Linear Differential Equations

Why Not Study Second Order Equations?

Damped Harmonic Oscillator	Swinging Pendulum
$mw''(t) + bw' + kw = 0$	$\theta''(t) + \frac{g}{L}\sin\theta(t) = 0$

Let $x = w$ and $y = w'$.

Then $x' = w' = y$ and $y' = w''$

so $mw''(t) + bw' + kw = 0$ becomes $my' + by + kx = 0$

Thus we have the system

$$x' = y$$

$$y' = -\frac{k}{m}x - \frac{b}{m}y$$

Let $x = \theta$ and $y = \theta'$. Then $\theta''(t) + \frac{g}{L}\sin\theta(t) = 0$ becomes system $x' = y, y' + \frac{g}{l}\sin x = 0$.

Systems of First Order Linear Differential Equations

$$x' = (\sin t)x + \left(\frac{1}{t}\right) y + 9z + 2t^3$$
$$y' = (t^2)x - (\cos 3t)y + (e^{-3t})z + \sec t$$
$$z' = (\log t)x - 2020y + (\tan t)z + e^{4t^2}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \sin t & \frac{1}{t} & 9 \\ t^2 & -\cos 3t & e^{-3t} \\ \log t & -2020 & \tan t \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2t^3 \\ \sec t \\ e^{4t^2} \end{pmatrix}$$

$$\mathbf{X'} = P(t)\, \mathbf{X} + \mathbf{g}(\mathrm{t})$$

Homogeneous: $\mathbf{X'} = P(t)\, \mathbf{X}$

Major Theorems On Systems of First Order Linear Differential Equations

Basic Existence and Uniqueness Result

THEOREM 6.2.1

(Existence and Uniqueness for First Order Linear Systems). If $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval $I = (\alpha, \beta)$, then there exists a unique solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where t_0 is any point in I, and $\mathbf{x}_0$ is any constant vector with n components. Moreover the solution exists throughout the interval I.

Linear Combinations of Solutions of Homogeneous Systems Are Solutions

THEOREM 6.2.2

(Principle of Superposition). If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are solutions of the homogeneous linear system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{5}$$

on the interval $I = (\alpha, \beta)$, then the linear combination

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

is also a solution of Eq. (5) on I.

Proof

Let $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$. The result follows from the linear operations of matrix multiplication and differentiation:

$$\begin{aligned}\mathbf{P}(t)\mathbf{x} &= \mathbf{P}(t)[c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k] \\ &= c_1\mathbf{P}(t)\mathbf{x}_1 + \cdots + c_k\mathbf{P}(t)\mathbf{x}_k \\ &= c_1\mathbf{x}_1' + \cdots + c_k\mathbf{x}_k' = \mathbf{x}'.\end{aligned}$$

Definition of Linear Independence

DEFINITION 6.2.3

The n vector functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are said to be **linearly independent on an interval** I if the only constants $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0} \tag{6}$$

for all $t \in I$ are $c_1 = c_2 = \cdots = c_n = 0$. If there exist constants $c_1, c_2, \ldots, c_n$, *not all zero*, such that Eq. (6) is true for all $t \in I$, the vector functions are said to be **linearly dependent** on I.

Jozef Maria Hoene Wronski
Józef Maria Hoene-Wroński
1776 –1853

Wronskians and the Struggle for Linear Independence

DEFINITION 6.2.4

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n solutions of the homogeneous linear system of differential equations $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ and let $\mathbf{X}(t)$ be the $n \times n$ matrix whose jth column is $\mathbf{x}_j(t)$, $j = 1, \ldots, n$,

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}. \tag{12}$$

The **Wronskian** $W = W[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ of the n solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is defined by

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det \mathbf{X}(t). \tag{13}$$

THEOREM 6.2.5

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an interval $I = (\alpha, \beta)$ in which $\mathbf{P}(t)$ is continuous.

(i) If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent on I, then $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) \neq 0$ at every point in I,

(ii) If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly dependent on I, then $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) = 0$ at every point in I.

Proof

Assume first that $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent on I. We then want to show that $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) \neq 0$ throughout I. To do this, we assume the contrary, that is, there is a point $t_0 \in I$ such that $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t_0) = 0$. This means that the column vectors $\{\mathbf{x}_1(t_0), \dots, \mathbf{x}_n(t_0)\}$ are linearly dependent (Theorem A.3.6) so that there exist constants $\hat{c}_1, \dots, \hat{c}_n$, not all zero, such that $\hat{c}_1\mathbf{x}_1(t_0) + \cdots + \hat{c}_n\mathbf{x}_n(t_0) = \mathbf{0}$. Then Theorem 6.2.2 implies that $\boldsymbol{\phi}(t) = \hat{c}_1\mathbf{x}_1(t) + \cdots + \hat{c}_n\mathbf{x}_n(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ that satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{0}$. The zero solution also satisfies the same initial value problem. The uniqueness part of Theorem 6.2.1 therefore implies that $\boldsymbol{\phi}$ is the zero solution, that is, $\boldsymbol{\phi}(t) = \hat{c}_1\mathbf{x}_1(t) + \cdots + \hat{c}_n\mathbf{x}_n(t) = \mathbf{0}$ for every $t \in (\alpha, \beta)$, contradicting our original assumption that $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent on I. This proves (i).

To prove (ii), assume that $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly dependent on I. Then there exist constants $\alpha_1, \dots, \alpha_n$, not all zero, such that $\alpha_1\mathbf{x}_1(t) + \cdots + \alpha_n\mathbf{x}_n(t) = \mathbf{0}$ for every $t \in I$. Consequently, for each $t \in I$, the vectors $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are linearly dependent. Thus $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) = 0$ at every point in I (Theorem A.3.6).

Dimension of Solution Space of x' = P(t) x

THEOREM 6.2.6

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be solutions of

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{14}$$

on the interval $\alpha < t < \beta$ such that, for some point $t_0 \in (\alpha, \beta)$, the Wronskian is nonzero, $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t_0) \neq 0$. Then each solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of Eq. (14) can be expressed as a linear combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$,

$$\boldsymbol{\phi}(t) = \hat{c}_1 \mathbf{x}_1(t) + \cdots + \hat{c}_n \mathbf{x}_n(t), \tag{15}$$

where the constants $\hat{c}_1, \dots, \hat{c}_n$ are uniquely determined.

Proof

Let $\boldsymbol{\phi}(t)$ be a given solution of Eq. (14). If we set $\mathbf{x}_0 = \boldsymbol{\phi}(t_0)$, then the vector function $\boldsymbol{\phi}$ is a solution of the initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{16}$$

By the principle of superposition, the linear combination $\boldsymbol{\psi}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ is also a solution of (14) for any choice of constants $c_1, \dots, c_n$. The requirement $\boldsymbol{\psi}(t_0) = \mathbf{x}_0$ leads to the linear algebraic system

$$\mathbf{X}(t_0)\mathbf{c} = \mathbf{x}_0, \tag{17}$$

where $\mathbf{X}(t)$ is defined by Eq. (12). Since $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t_0) \neq 0$, the linear algebraic system (17) has a unique solution (see Theorem A.3.7) that we denote by $\hat{c}_1, \dots, \hat{c}_n$. Thus the particular member $\hat{\boldsymbol{\psi}}(t) = \hat{c}_1\mathbf{x}_1(t) + \cdots + \hat{c}_n\mathbf{x}_n(t)$ of the n-parameter family represented by $\boldsymbol{\psi}(t)$ also satisfies the initial value problem (16). By the uniqueness part of Theorem 6.2.1, it follows that $\boldsymbol{\phi} = \hat{\boldsymbol{\psi}} = \hat{c}_1\mathbf{x}_1 + \cdots + \hat{c}_n\mathbf{x}_n$. Since $\boldsymbol{\phi}$ is arbitrary, the result holds (with different constants, of course) for every solution of Eq. (14).

THEOREM 6.2.7

Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix};$$

further let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ that satisfy the initial conditions

$$\mathbf{x}_1(t_0) = \mathbf{e}_1, \quad \ldots, \quad \mathbf{x}_n(t_0) = \mathbf{e}_n,$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ form a fundamental set of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Homogenous Linear Systems
With Constant Coefficients

X' $= P(\mathrm{t})$ **X** where $P(\mathrm{t})$ is a matrix of CONSTANTS

X' $= A$ **X** where A is an $n \times n$ matrix of CONSTANTS

$$
\begin{aligned}
x' &= 5x + 29y - 4z - 1w \\
y' &= 12x + 21y - 19z + 66w \\
z' &= -8x + 15y + 7z - 2w \\
w' &= 4x + 9y + 20z + 20w
\end{aligned}
$$

Linear Systems with Constant Coefficients

Simplest Case

THEOREM 6.3.1

Let $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ be eigenpairs for the real, $n \times n$ constant matrix A. Assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are real and that the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Then

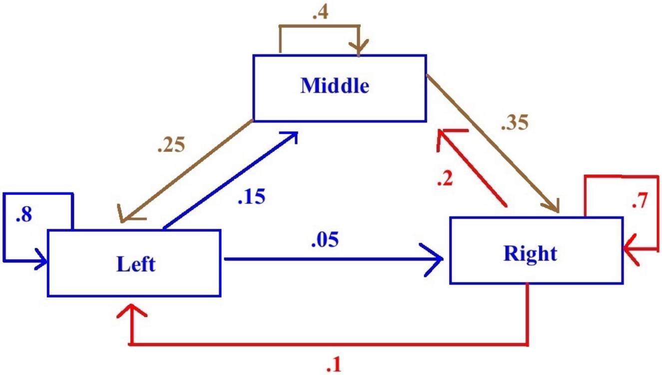
$$\left\{e^{\lambda_1 t}\mathbf{v}_1, \ldots, e^{\lambda_n t}\mathbf{v}_n\right\} \tag{6}$$

is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on the interval $(-\infty, \infty)$. The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is therefore given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n, \tag{7}$$

where $c_1, \ldots, c_n$ are arbitrary constants.

A Differential Equations Model of Political Movement



$$L' = -.2L + .25M + .1R$$
$$M' = .15L - .6M + .2R$$
$$R' = .05L + .35M - .3R$$

Consider a system of first order linear homogeneous differential equations with constant coefficients

$$\mathbf{X'} = A\ \mathbf{X}$$

where A is $n \times n$ matrix of constants and $\mathbf{X}$ is $n \times 1$ vector of functions of t.

Theorem 1 If λ is an eigenvalue of A with corresponding eigenvector v, then $e^{\lambda t} v$ is a solution of X' = AX.

Proof: If $\mathbf{X} = e^{\lambda t}\vec{v}$, then

$$\begin{aligned}\mathbf{X}\ ' &= \lambda e^{\lambda t}\vec{v} \\ &= e^{\lambda t}\lambda\vec{v} \\ &= e^{\lambda t}\ A\ \vec{v} \\ &= A\ e^{\lambda t}\vec{v} \\ &= A\ \mathbf{X}\end{aligned}$$

Theorem 2 If λ and μ are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$
(that is, A $\vec{v} = \lambda\ \vec{v}$ and A$\vec{w} = \mu\vec{w}$)
then

1. $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}\}$ is a linearly independent set of solutions of **X'** = A**X**

Proof of 1: Suppose C1 and C2 are constants such that
(*) C1 $\vec{v}$ + C2 $\vec{w} = \vec{0}$.
Multiply (*) by A to obtain (**) C1 $\lambda\ \vec{v}$ + C2 $\mu\ \vec{w} = \vec{0}$
Multiply (*) by μ to obtain (***) C1 $\mu\ \vec{v}$ + C2 $\mu\ \vec{w} = \vec{0}$
Subtract (***) from (**)to obtain C1(λ - μ) $\vec{v} = \vec{0}$
But λ - $\mu \neq 0$ and $\vec{v} \neq 0$; Hence C1 = 0
which implies C2 $\vec{w} = \vec{0}$ and that implies C2 = 0.

Theorem 2 If λ and μ are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then

1. $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}\}$ is a linearly independent set of solutions of **X'** $=$ A**X**

Proof of 2: Suppose C1 and C2 are constants such that

$$C1\ e^{\lambda t}\vec{v} + C2\ e^{\mu t}\vec{w} = \vec{0}.$$

Evaluate both sides at $t = 0$:

$$C1\ e^{\lambda 0}\vec{v} + C2\ e^{\mu 0}\vec{w} = \vec{0}$$

$$C1\ e^{0}\vec{v} + C2\ e^{0}\vec{w} = \vec{0}$$

$$C1\ \vec{v} + C2\ \vec{w} = \vec{0}$$

which implies C1 and C2 are both 0.

A Generalization of Theorem 2

Theorem 3 If λ , μ and α are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v}$, $\vec{w}$ and $\vec{u}$
(that is, A $\vec{v} = \lambda\ \vec{v}$, A$\vec{w} = \mu\vec{w}$, A$\vec{u} = \alpha\vec{u}$)
then

1. $\{\vec{v}, \vec{w}, \vec{u}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}, e^{\alpha t}\vec{u}\}$ is a linearly independent set of solutions of **X'** $=$ A**X**

A Even Bigger Generalization of Theorem 2

Theorem 4 If $\lambda_1, \lambda_2, ..., \lambda_k$, are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_k}$
(that is, A $\vec{v_i} = \lambda_i$ for each $i = 1, 2, 3, ..., k$
then

1. $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_k}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda_1 t}\vec{v_1}, e^{\lambda_2 t}\vec{v_2}, ..., e^{\lambda_k t}\vec{v_k}\}$ is a linearly independent set of solutions of **X'** = A**X**