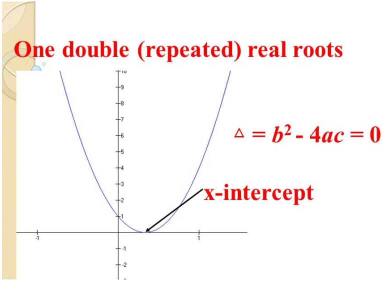
MATH 226: Differential Equations



Class 19: October 26, 2022

Examples of Repeated Eigenvalues
Chapter 3: Summary of Results
Assignment 13

Announcements

- Presentation Tomorrow: "Does That Map Men What You Think It Means?"
  Peter Gao, University of Washington
  12:30 – 1:30 in Warner 101
- Friday's Class on Zoom
  https://middlebury.zoom.us/j/8328362601?pwd=cmtmS1FzWTY5N
- Second Project Due Friday, November 4
- Exam 2: Wednesday, November 16

Theorem: If λ and μ are distinct eigenvalues (real or complex) of a 2×2 matrix A having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x}' = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Possibilities

2 Real Unequal Roots

2 Complex Roots

1 Real Double Root

Complex Eigenvalue

$$\lambda = u + iv$$

Solutions Look Like

$$e^{ut}(\mathbf{a}\cos vt - \mathbf{b}\sin vt) \text{ and } e^{ut}(\mathbf{a}\sin vt + \mathbf{b}\cos vt)$$

where $\mathbf{a} + i\mathbf{b}$ is an eigenvector

Long Term Qualitative Behavior depends on sign of u

$u > 0$	Spiral Source
$u < 0$	Spiral Sink
$u = 0$	Center

Double Roots

Begin With Some Very Simple Systems of form X' = AX

Example 1: $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ so system is uncoupled $\begin{aligned} x' &= ax \\ y' &= ay \end{aligned}$

which has solution

$$\begin{aligned} x &= C_1 e^{at} \\ y &= C_2 e^{at} \end{aligned} \rightarrow \vec{X} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \left[C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{at}$$

From Eigenvalue/Eigenvector Perspective:

$$A - \lambda I = \begin{pmatrix} a - \lambda & 0 \\ 0 & a - \lambda \end{pmatrix}; \ \det(A - \lambda I) = (a - \lambda)^2 \text{ so } \lambda = a \text{ is double root}$$

For Eigenvectors: $\lambda = a$:

$$A - \lambda I = A - aI = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Any nonzero vector is an eigenvector. We can choose

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eigenvalue/Eigenvector Perspective:

$$A-\lambda I = \begin{pmatrix} a-\lambda & 0 \\ 0 & a-\lambda \end{pmatrix}; \ \det(A-\lambda I) = (a-\lambda)^2 \text{ so } \lambda = a \text{ is double root}$$

The **Algebraic Multiplicity** is 2

For Eigenvectors: $\lambda = a$:

$$A - \lambda I = A - aI = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Any nonzero vector is an eigenvector. We can choose $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ The **Geometric Multiplicity** is also 2.

Double Roots

Example 2: $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ System is $\begin{aligned} x' &= ax + 1y \\ y' &= ay \end{aligned}$

Solve $y' = ay$ to get $y = C_2 e^{at}$
Then $x' = ax + y = ax + C_2 e^{at}$ which we can rewrite as
$x' - ax = C_2 e^{at}$.
This is a first-order Linear Differential Equation.
An Integrating Factor is e^{-at} giving

$$x' e^{-at} - axe^{-at} = C_2 e^{at} e^{-at} = C_2$$

so $(xe^{-at})' = C_2$

implying $xe^{-at} = C_2 t + C_1$

or $x = C_1 e^{at} + C_2 te^{at}$

Example 2: $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ System is $\begin{aligned} x' &= ax + 1y \\ y' &= ay \end{aligned}$

$$y = C_2 e^{at}, x = C_1 e^{at} + C_2 t e^{at}$$

We can write the solution of the original system as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 e^{at} + C_2 t e^{at} \\ C_2 e^{at} \end{pmatrix}$$

$$= e^{at} \begin{pmatrix} C_1 \\ C2 \end{pmatrix} + t e^{at} \begin{pmatrix} C_2 \\ 0 \end{pmatrix}$$

Note new type of solution te^{at}

From Eigenvalue/Eigenvector Perspective:

$$A - \lambda I = \begin{pmatrix} a - \lambda & 1 \\ 0 & a - \lambda \end{pmatrix}; \det(A - \lambda I) = (a - \lambda)^2 \text{ so } \lambda = a \text{ is double root}$$

For Eigenvectors: $\lambda = a$:

$$A - \lambda I = A - aI = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

From Eigenvalue/Eigenvector Perspective:

$A-\lambda I = \begin{pmatrix} a-\lambda & 1 \\ 0 & a-\lambda \end{pmatrix}$; $\det(A-\lambda I) = (a-\lambda)^2$ so $\lambda = a$ is double root

The **Algebraic Multiplicity** is 2
For Eigenvectors: $\lambda = a$:

$$A - \lambda I = A - aI = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ means } v_2 = 0, v_1 = \text{ any value.}$$

We may choose $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

One Solution of X'=AX is $e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Can We Produce a Solution of the Form

$$\vec{X} = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$$

where $\vec{v}$ is an eigenvector of A associated with λ?
If so, how do we find $\vec{w}$?
We need $\vec{X'} = A\vec{v}$

Left Hand Side: $\vec{X'} = t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{w}.$

$$= e^{\lambda t}\left[t\lambda\vec{v} + \vec{v} + \lambda\vec{w}\right]$$

Right Hand Side: $$\begin{aligned} A\vec{X} &= A(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}) \\ &= te^{\lambda t}A\vec{v} + e^{\lambda t}A\vec{w} \\ &= e^{\lambda t}\left[tA\vec{v} + A\vec{w}\right] \text{ but } A\vec{v} = \lambda\vec{v} \\ &= e^{\lambda t}\left[t\lambda\vec{v} + A\vec{w}\right] \end{aligned}$$

We need $\vec{X'} = A\vec{v}$

$$\begin{aligned} \vec{X'} &= e^{\lambda t}\left[t\lambda\vec{v} + \vec{v} + \lambda\vec{w}\right] \\ A\vec{X} &= e^{\lambda t}\left[t\lambda\vec{v} + A\vec{w}\right] \end{aligned}$$

To Get Equality, We Need Entries in [] To Match:

$$\begin{aligned} t\lambda\vec{v} + \vec{v} + \lambda\vec{w} &= t\lambda\vec{v} + A\vec{w} \\ A\vec{w} &= \vec{v} + \lambda\vec{w} \\ .\text{ or } A\vec{w} - \lambda\vec{w} &= \vec{v} \\ (A - \lambda I)\vec{w} &= \vec{v} \end{aligned}$$

We can solve this algebraic equation for $\vec{w}$

To get a solution of the form

$$\vec{X} = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w} \text{ to } \vec{X'} = AX$$

choose $\vec{v}$ to be an eigenvector associated with eigenvalue λ and $\vec{w}$ to satisfy $(A - \lambda I)\vec{w} = \vec{v}$

Claim: $\{\vec{v}, \vec{w}\}$ is a Linearly Independent Set
Proof: Suppose $C_1\vec{v} + C_2\vec{w} = \vec{0}$
Multiply both sides by $(A - \lambda I)$
$C_1(A - \lambda I)\vec{v} + C_2(A - \lambda I)\vec{w} = (A - \lambda I)\vec{0}$
But $(A - \lambda I)\vec{v} = \vec{0}, (A - \lambda I)\vec{0} = \vec{0}, (A - \lambda I)\vec{w} = \vec{v}$
So $C_2\vec{v} = \vec{0}$ BUT $\vec{v} \neq \vec{0}$ so $C_2 = 0$
Hence $C_1\vec{v} = \vec{0}$ but this again yields $C_1 = 0$.

$\{\vec{v}, \vec{w}\}$ is a Linearly Independent Set

From here, it is easy to show that

$\{e^{\lambda t}\vec{v}, te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}\}$ is a Linearly Independent Set of Solutions

Proof: Suppose $C_1 e^{\lambda t}\vec{v} + C_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}) = \vec{0}$

Evaluate at $t = 0$, using $e^{\lambda 0} = 1$:

$$C_1\vec{v} + C_2(\vec{0} + \vec{w}) = \vec{0}$$

$$C_1\vec{v} + C_2\vec{w} = \vec{0}$$

which implies $C_1 = 0, C_2 = 0$

$$A = \begin{pmatrix} 4 & b \\ -2 & 6 \end{pmatrix} \text{ gives } A - \lambda I = \begin{pmatrix} 4-\lambda & b \\ -2 & 6-\lambda \end{pmatrix}$$

$$\begin{aligned} det(A - \lambda I) &= (4-\lambda)(6-\lambda) - (-2)b \\ &= 24 - 10\lambda + \lambda^2 + 2b \\ &= \lambda^2 - 10\lambda + 24 + 2b \end{aligned}$$

$$\begin{aligned} \text{Eigenvalues: } \lambda &= \frac{10 \pm \sqrt{100 - 4(24 + 2b)}}{2} \\ &= \frac{10 \pm \sqrt{4 - 4(2b)}}{2} \\ &= \frac{10 \pm 2\sqrt{1 - 2b}}{2} \\ &= 5 \pm \sqrt{1 - 2b} \end{aligned}$$

Set $b = 1/2$ so $\lambda = 5$ is a double root.

$$A = \begin{pmatrix} 4 & 1/2 \\ -2 & 6 \end{pmatrix} \text{ gives } A - \lambda I = \begin{pmatrix} 4-\lambda & 1/2 \\ -2 & 6-\lambda \end{pmatrix}$$

which has $\lambda = 5$ as a double root (Algebraic Multiplicity $= 2$)

To Find Eigenvectors: $A - \lambda I = A - 5I = \begin{pmatrix} -1 & 1/2 \\ -2 & 1 \end{pmatrix}$

which row reduces to $\begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$

$(A - \lambda I)\vec{v} = \vec{0}$ becomes $-2v_1 + v_2 = 0$ so $v_2 = 2v_1$ or $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The geometric multiplicity is 1

One solution to $\vec{X}' = A\vec{X}$ is $e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

To get another solution, solve $(A - \lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix} -1 & 1/2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

One solution to $\vec{X}' = A\vec{X}$ is $e^{5t}\begin{pmatrix}1\\2\end{pmatrix}$

To get another solution, solve $(A - \lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix}-1 & 1/2\\-2 & 1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

We get two equations: $-w_1 + \frac{1}{2}w_2 = 1, -2w_1 + w_2 = 2$

which yield $w_2 = 2 + 2w_1$

We can choose $\vec{w} = \begin{pmatrix}1\\4\end{pmatrix}$

Another solution to $\vec{X}' = A\vec{X}$ is $te^{5t}\vec{v} + e^{5t}\vec{w} =$

$$te^{5t}\begin{pmatrix}1\\2\end{pmatrix} + e^{5t}\begin{pmatrix}1\\4\end{pmatrix}$$

We can write general solution as

$$C_1e^{5t}\begin{pmatrix}1\\2\end{pmatrix} + C_2\left[te^{5t}\begin{pmatrix}1\\2\end{pmatrix} + e^{5t}\begin{pmatrix}1\\4\end{pmatrix}\right]$$

General solution to $\vec{X'} = A\vec{X}$ is

$$C_1 e^{5t}\begin{pmatrix}1\\2\end{pmatrix} + C_2\left[te^{5t}\begin{pmatrix}1\\2\end{pmatrix} + e^{5t}\begin{pmatrix}1\\4\end{pmatrix}\right] = e^{5t}\begin{pmatrix}C_1 + C_2 t + C_2\\2C_1 + 2C_2 t + 4C_2\end{pmatrix}$$

With Initial Condition $X(\vec{0}) = \begin{pmatrix}4\\6\end{pmatrix}$, we have

$$C_1 + C_2 = 4$$
$$2C_1 + 4C_2 = 6$$

which has solution: $C_1 = 5, C_2 = -1$ yielding

$$e^{5t}\begin{pmatrix}5 - t - 1\\10 - 2t - 4\end{pmatrix} = e^{5t}\begin{pmatrix}4 - t\\6 - 2t\end{pmatrix}$$

Chapter 3: Summary of Results from Brannan and Boyce

This completes our investigation of the possible behavior of solutions of a two-dimensional linear homogeneous system with constant coefficients, $\mathbf{x}'=A\mathbf{x}$. When the coefficient matrix A has a nonzero determinant, there is a single equilibrium solution, or critical point, which is located at the origin. By reflecting on the possibilities explored in this section and in the two preceding ones, and by examining the corresponding figures, we can make several observations:

1. After a long time, each individual trajectory exhibits one of only three types of behavior. As $t \to \infty$, each trajectory becomes unbounded, approaches the critical point $\mathbf{x} = \mathbf{0}$, or repeatedly traverses a closed curve, corresponding to a periodic solution, that surrounds the critical point.

2. Viewed as a whole, the pattern of trajectories in each case is relatively simple. To be more specific, through each point (x_0, y_0) in the phase plane there is only one trajectory; thus the trajectories do not cross each other. Do not be misled by the figures, in which it sometimes appears that many trajectories pass through the critical point $\mathbf{x} = \mathbf{0}$. In fact, the only solution passing through the origin is the equilibrium solution $\mathbf{x} = \mathbf{0}$. The other solutions that appear to pass through the origin actually only approach this point as $t \to \infty$ or $t \to -\infty$.

3. In each case, the set of all trajectories is such that one of three situations occurs.

a. All trajectories approach the critical point $\mathbf{x} = \mathbf{0}$ as $t \to \infty$. This is the case if the eigenvalues are real and negative or complex with a negative real part. The origin is either a nodal or a spiral sink.

b. All trajectories remain bounded but do not approach the origin as $t \to \infty$. This is the case if the eigenvalues are purely imaginary. The origin is a center.

c. Some trajectories, and possibly all trajectories except $\mathbf{x} = \mathbf{0}$, become unbounded as $t \to \infty$. This is the case if at least one of the eigenvalues is positive or if the eigenvalues have a positive real part. The origin is a nodal source, a spiral source, or a saddle point.

Stability properties of linear systems $\mathbf{x}' = A\mathbf{x}$ with $\det(A - \lambda I) = 0$ and $\det A \neq 0$.

Eigenvalues	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_1, \lambda_2 = \mu \pm i\nu$	Spiral point	
$\mu > 0$		Unstable
$\mu < 0$		Asymptotically stable
$\lambda_1 = i\nu, \lambda_2 = -i\nu$	Center	Stable