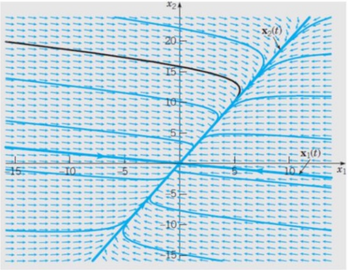
MATH 226: Differential Equations

Nodal Sources and Nodal Sinks



The pattern of trajectories in Figure is typical of all second order systems **x'** = **Ax** whose eigenvalues are real, different, and of the same sign. The origin is called a **node** for such a system.

Class 16: October 19, 2022

MidTerm Feedback Form

"Nodal Links," A Maple print out on Unequal Roots
Unequal Roots.mw in Handouts Folder for Maple File

Modernized MATLAB apps for Phase Planes (systems of differential equations) and Slope Fields (single differential equation)

URL at https://www.mathworks.com/matlabcentral/fileexchange/91705-phase-plane-and-slope-field-apps

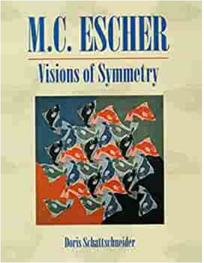
Announcements

For Next Time, Work Through

Complex Numbers

- Use the imaginary unit *i* to write complex numbers, and add, subtract, and multiply complex numbers.
- Find complex solutions of quadratic equations.
- Write the trigonometric forms of complex numbers.
- Find powers and *n*th roots of complex numbers.

Mathematician of the Week : **Doris Schattschneider**



October 19, 1939 –

Doris Schattschneider is an American mathematician who became professor of mathematics at the Moravian College. She was the first female editor of Mathematics Magazine and won many awards for her excellence in expository writing.

Biography

Our Main Agenda

Solve **X' = A X** where A is an $n \times n$ matrix of constants and **X** is an n-dimensional vector of functions.

Results So Far

Theorem: The set of solutions is a vector space.

We can find some solutions of the form $e^{\lambda t}\vec{v}$ where λ is an eigenvalue of A and $\vec{v}$ is an associated eigenvector.

Distinct eigenvalues give rise to linearly independent solutions.

A linearly independent set of n solutions is a basis for the vector space.

Outstanding Questions

What To Do If 0 is an eigenvalue?

How to handle complex eigenvalues.

How to find n linearly independent solutions to **X' = A X** when there are not enough of the form $e^{\lambda t}\vec{v}$. .

Today:

Continue Study of Linear Homogeneous Systems With Constant Coefficients

X' = A X

2×2 Case

With Real, Unequal Eigenvalues

Theorem: If λ and μ are distinct eigenvalues (real or complex) of a 2×2 matrix A having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x}' = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Characteristic Polynomial of A is det(A - λ I) =

$$\lambda^2 - (a+d)\lambda + ad - bc$$

λ^2 - Trace(A) λ + Det A

Characteristic Equation: det(A - λ I) = 0

Eigenvalues Are Roots of Characteristic Polynomial

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Possibilities

2 Real Unequal Roots

2 Complex Roots

1 Real Double Root

More About 2 Real Unequal Roots Case

$$\mathbf{x}' = A\,\mathbf{x}$$

The Origin (0,0) is an equilibrium and is called a **NODE**

More About 2 Real Unequal Roots Case $\mathbf{x}' = A\,\mathbf{x}$

The Origin (0,0) is an equilibrium and is called a **NODE**

$\lambda_1, \lambda_2 < 0$ Node is Asymptotically Stable
NODAL SINK

$\lambda_1, \lambda_2 > 0$ Node is Unstable
NODAL SOURCE

Opposite Sign Node is Unstable
SADDLE POINT

Nodal Sink $\begin{bmatrix} -7 & 3 \\ 2 & -2 \end{bmatrix}$ $\lambda = -1, -8$

Nodal Source $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ $\lambda = 3, 2$

Saddle Point $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\lambda = 3, -1$

ZERO AS AN EIGENVALUE

Example: Richardson Arms Race Model
With Parallel Stable Lines

$$x' = -mx + ay + r$$
$$y' = bx - ny + s$$

Slope of **L** $= \frac{m}{a}$
Slope of **L'** $= \frac{b}{n}$

$$\text{Parallel if } \frac{m}{a} = \frac{b}{n}$$

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix}$$
$$mn = ab \Leftrightarrow mn - ab = 0 \Leftrightarrow det(A) = 0$$

Characteristic Equation: $\lambda^2 + (m+n)\lambda + (mn - ab) = 0$

$$\lambda^2 + (m+n)\lambda = 0$$
$$\lambda(\lambda + (m+n)) = 0 \Rightarrow \lambda = 0, \lambda = -(m+n)$$

ZERO AS AN EIGENVALUE EXAMPLE

$$m = 3, a = 6, n = 8, b = 4$$

$$A = \begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix}$$

$$\det \begin{bmatrix} -3-\lambda & 6 \\ 4 & -8-\lambda \end{bmatrix}$$

$$= (-3-\lambda)(-8-\lambda) - 24 = \lambda^2 + 11\lambda + 24 - 24$$

$$= \lambda^2 + 11\lambda = \lambda(\lambda + 11)$$

$$\lambda = 0, \lambda = -11$$

For $\lambda = -11$:

$$\begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4v_1 + 3v_2 = 0$$

$$\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

For $\lambda = 0$:

$$\begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3w_1 + 6w_2 = 0$$

$$\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = -11, \vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\lambda = 0, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

General Solution To

$$x' = -3x + 6y$$

$$y' = 4x = 8y$$

is $\mathbf{x} = C_1 e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-11t} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$x = 2C_1 - 3C_2 e^{-11t}$$

$$y = C_1 + 4C_2 e^{-11t}$$

Particular Solution: (x_0, y_0) at $t = 0$

$$x_0 = 2C_1 - 3C_2$$
$$y_0 = C_1 + 4C_2$$

$2C_1 - 3C_2 = x_0$	$8C_1 - 12C_2 = 4x_0$
$-2C_1 - 8C_2 = -2y_0$	$3C_1 + 12C_2 = 3y_0$
Add Equations	Add Equations
$-11C_2 = x_0 - 2y_0$	$11C_1 = 4x_0 + 3y_0$

$$C_1 = \frac{4x_0 + 3y_0}{11}, \; C_2 = \frac{-x_0 + 2y_0}{11}$$

$$x = 2\left(\frac{4x_0 + 3y_0}{11}\right) - 3\left(\frac{-x_0 + 2y_0}{11}\right) e^{-11t}$$

$$y = \frac{4x_0 + 3y_0}{11} + 4\left(\frac{-x_0 + 2y_0}{11}\right) e^{-11t}$$

Next Time

Complex Eigenvalues