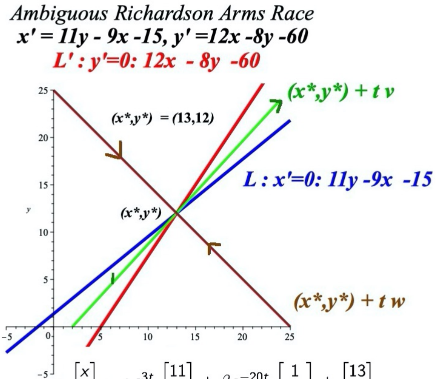
MATH 226: Differential Equations



$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + \beta e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 13 \\ 12 \end{bmatrix}$$

Class 15: October 17, 2022

Notes on Exam 1

Notes on Assignment 8

Assignment 9

In Handouts Folder:

Unequal Roots.mw

Problem 30 on Page 144.maple

PhasePlane Tutorial (also need PlotPhasePlane.m)

Solving Linear System in MATLAB

Announcements

For Next Time, Work Through

Complex Numbers

- Use the imaginary unit *i* to write complex numbers, and add, subtract, and multiply complex numbers.
- Find complex solutions of quadratic equations.
- Write the trigonometric forms of complex numbers.
- Find powers and *n*th roots of complex numbers.

Comments/Questions on Exam 1

Problem 1: $Q' = r(Q - R)$

$Q = T$ is Equilibrium Solution

Can Q **reach** this value?

Comments on Some Homework Problems

Section 3.1 : Exercise 38

Show that $\lambda = 0$ is an eigenvalue for a matrix $\mathbf{A}$

if and only if

$\det(\mathbf{A}) = 0$.

Must Prove Both:

- If $\lambda = 0$ is an eigenvalue, then $\det(\mathbf{A}) = 0$, **AND**
- If $\det(\mathbf{A}) = 0$, then $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

If $\lambda = 0$ is an eigenvalue, then det(**A**) $=$0

Proof:
If $\lambda = 0$ is an eigenvalue, then there is a nonzero vector $\vec{v}$ such that **A** $\vec{v} = \vec{0}$.
Hence **A** is not invertible
so det(**A**) $= 0$.

If det(**A**) $=$0, then $\lambda = 0$ is an eigenvalue of **A**.

Proof: Note that **A** $=$ **A** - **0** $=$ **A** - 0 **I**.
If det(**A**) $= 0$, then det(**A** - 0 **I**) $= 0$
so 0 satisfies the characteristic equation for **A** and hence is an eigenvalue.

Comments on Some Homework Problems

Section 3.2 Exercise 30c

Use a computer to draw component plots of the initial value problem and the equilibrium solutions.

See *Problem 30 on Page 144.maple*

Major Goal:

Understand Systems of Differential Equations

$$X' = P(t)\,X + g(t)$$

where P(t) is n $\times$ n Matrix of Functions

Existence and Uniqueness Theorems for Linear Systems

Theorem 2.4.1: If $p(t)$ and $g(t)$ are continuous functions on an open interval I containing the point $t = t_o$ and y_o is any prescribed initial value, then there exists a unique solution $y = \phi(t)$ of the differential equation that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for all t in I with $\phi(t_o) = y_o$.

Theorem 3.2.1: If $P(t)$ is an $n \times n$ matrix and $\mathbf{g}(t)$ is an $n \times 1$ vector whose entries are continuous on an open interval I containing the point t_o and $\mathbf{y_o}$ is any prescribed initial value, then there is a unique solution $\mathbf{y} = \mathbf{\Phi}(t)$ of the system of differential equations

$$\boldsymbol{X}' = P(t)\boldsymbol{X} + \mathbf{g}(t)$$

for all t in I with $\mathbf{\Phi}(t_o) = \mathbf{y_o}$.

Theorem 3.2.1: If $P(t)$ is an $n \times n$ matrix and $\mathbf{g}(t)$ is an $n \times 1$ vector whose entries are continuous on an open interval I containing the point t_o and $\mathbf{y_0}$ is any prescribed initial value, then there is a unique solution $\mathbf{y} = \mathbf{\Phi}(t)$ of the system of differential equations

$$\boldsymbol{X}' = P(t)\boldsymbol{X} + \mathbf{g}(t)$$

for all t in I with $\mathbf{\Phi}(t_o) = \mathbf{y_0}$.

What Does This Theorem Say in the case $P(t)$ is an $n \times n$ matrix of **constants** and $\mathbf{g(t)}$ is identically 0?

There is a unique solution valid for all real numbers!

Focus on Linear Homogeneous System with Constant Coefficients

$$\mathbf{X'} = A\,\mathbf{X}$$

where A is a 2×2 matrix.

Begin with Earlier Example

$$x' = -9x + 11y$$
$$y' = 12x - 8y$$

$$A = \begin{bmatrix} -9 & 11 \\ 12 & -8 \end{bmatrix}$$

$$\mathbf{X'} = A\,\mathbf{X}$$

where

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix} \text{ has solution}$$

$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w}$$

where α and β are arbitrary constants
λ is an eigenvalue of A with associated eigenvector $\vec{v}$ and
$\mu \neq \lambda$ is an eigenvalue of A with associated eigenvector $\vec{w}$.

The solution of the original system is then

$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w} + \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

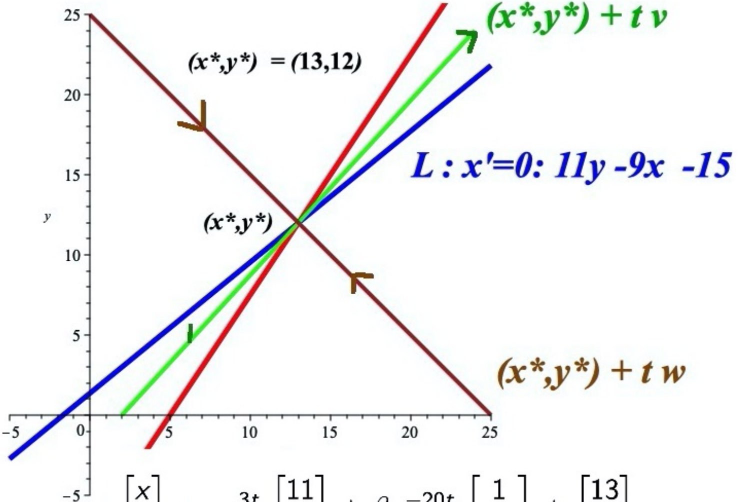
Two Particular Examples: of Richardson Arms Race Model

$x' = -5x + 4y + 1$ $y' = 3x - 4y + 2$	$x' = 11y - 9x - 15$ $y' = 12x - 8y - 60$
$(x^*, y^*) = (\frac{3}{2}, \frac{13}{8})$ $A = \begin{bmatrix} -5 & 4 \\ 3 & -4 \end{bmatrix}$ $\lambda = -1, \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mu = -8, \vec{w} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$	$(x^*, y^*) = (13, 12)$ $A = \begin{bmatrix} -9 & 11 \\ 12 & -8 \end{bmatrix}$ $\lambda = 3, \vec{v} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$ $\mu = -20, \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\alpha e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{-8t} \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ \frac{13}{8} \end{bmatrix}$	$\alpha e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + \beta e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 13 \\ 12 \end{bmatrix}$

Ambiguous Richardson Arms Race

$$x' = 11y - 9x - 15, \quad y' = 12x - 8y - 60$$

$$L' : y' = 0 : 12x - 8y - 60$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + \beta e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 13 \\ 12 \end{bmatrix}$$

Two solutions to the homogeneous system are

$$e^{3t}\vec{v} \text{ and } e^{-20t}\vec{w}$$

$$e^{3t}\begin{bmatrix}11\\12\end{bmatrix} \text{ and } e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$$

Then $C_1e^{3t}\begin{bmatrix}11\\12\end{bmatrix}+C_2e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$ is a solution for any constants C_1 and C_2.

Now suppose $\Phi(t)$ is any solution to the system with $\Phi(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$

CLAIM: We can find $C1$ and $C2$ so that

$$\Phi(t)=C_1e^{3t}\begin{bmatrix}11\\12\end{bmatrix}+C_2e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$$

CLAIM: We can find $C1$ and $C2$ so that

$$\Phi(t) = C_1 e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

NEED: Agreement at $t = 0$:

$$C_1 e^{3\times 0} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 e^{-20\times 0} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$C_1 \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$11C_1 + 12C2 = x_0$$
$$12C_1 - 1C_2 = y_0$$

$$\begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

has a solution for all x_0, y_0 exactly when the coefficient matrix

$M = \begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix}$ is invertible

and this happens if and only the columns of the coefficient matrix are a linearly independent set of vectors.

But the columns are $\vec{v}$ and $\vec{w}$ which are eigenvectors belonging to distinct eigenvalues

so they do form a linearly independent set.

The solution will be

$$\begin{bmatrix} C1 \\ C2 \end{bmatrix} = M^{-1} \begin{bmatrix} x_0 \\ y_o \end{bmatrix} = \frac{-1}{23} \begin{bmatrix} -1 & -1 \\ -12 & 11 \end{bmatrix} \begin{bmatrix} x_0 \\ y_o \end{bmatrix}$$

The solution will be

$$\begin{bmatrix} C1 \\ C2 \end{bmatrix} = M^{-1} \begin{bmatrix} x_0 \\ y_o \end{bmatrix} = \frac{-1}{23} \begin{bmatrix} -1 & -1 \\ -12 & 11 \end{bmatrix} \begin{bmatrix} x_0 \\ y_o \end{bmatrix}$$

$$\text{Thus } \begin{bmatrix} C1 \\ C2 \end{bmatrix} = \begin{bmatrix} \frac{x_0+y_0}{23} \\ \frac{12x_0-11y_0}{23} \end{bmatrix}.$$

We have found one solution of the homogeneous system that

- Agrees with Φ and $t = 0$ and
- Is a linear combination of $e^{3t}\vec{v}$ and $e^{-20t}\vec{w}$.

By The Uniqueness Theorem, Φ **must** be a linear combination of these two solutions.

Thus these two particular solutions are a **Spanning Set** for the collection of all solutions to the homogeneous system.

The two particular solutions $e^{3t}\vec{v}$ and $e^{-20t}\vec{w}$ form a **Spanning Set** for the collection of all solutions to the homogeneous system.

What Made This Work?

$\vec{v}, \vec{w}$ is a linearly independent set of vectors which we know is true since they are associated with two distinct eigenvalues.

Moreover, the two solutions themselves are Linearly Independent Solutions. They form a **BASIS** for the set of all solutions to the homogeneous system of differential equations $\mathbf{X'} = A\,\mathbf{X}$.

Theorem: Let λ and μ be distinct eigenvalues for a square matrix A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$. Then $e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}$ is a linearly independent set of solutions for $\mathbf{X'} = A\,\mathbf{X}$.

Our Main Agenda

Solve **X' = A X** where A is an $n \times n$ matrix of constants and **X** is an n-dimensional vector of functions.

Results So Far

Theorem: The set of solutions is an n-dimensional vector space.

We can find some solutions of the form $e^{\lambda t}\vec{v}$ where λ is an eigenvalue of A and $\vec{v}$ is an associated eigenvector.

Distinct eigenvalues give rise to linearly independent solutions.

Outstanding Questions

How to handle complex eigenvalues.

How to find n linearly independent solutions to **X' = A X** when there are not enough of the form $e^{\lambda t}\vec{v}$. .

Current Goal:

Continue Study of Linear Homogeneous Systems With Constant Coefficients

X' = A X

2×2 Case

Theorem: If λ and μ are distinct eigenvalues (real or complex) of a 2×2 matrix A having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x}' = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Characteristic Polynomial of A is det(A - λ I) =

$$\lambda^2 - (a+d)\lambda + ad - bc$$

λ^2 - Trace(A) λ + Det A

Characteristic Equation: det(A - λ I) = 0

Eigenvalues Are Roots of Characteristic Polynomial

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Possibilities

2 Real Unequal Roots

2 Complex Roots

1 Real Double Root

More About 2 Real Unequal Roots Case

$$\mathbf{x'} = A\,\mathbf{x}$$

The Origin (0,0) is an equilibrium and is called a **NODE**

More About 2 Real Unequal Roots Case $\mathbf{x}' = A\,\mathbf{x}$

The Origin (0,0) is an equilibrium and is called a **NODE**

$\lambda_1, \lambda_2 < 0$	Node is Asymptotically Stable **NODAL SINK**
$\lambda_1, \lambda_2 > 0$	Node is Unstable **NODAL SOURCE**
Opposite Sign	Node is Unstable **SADDLE POINT**

Nodal Sink $\begin{bmatrix} -7 & 3 \\ 2 & -2 \end{bmatrix}$ $\lambda = -1, -8$

Nodal Source $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ $\lambda = 3, 2$

Saddle Point $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\lambda = 3, -1$

ZERO AS AN EIGENVALUE

Example: Richardson Arms Race Model
With Parallel Stable Lines

$$x' = -mx + ay + r$$
$$y' = bx - ny + s$$

Slope of **L** $= \frac{m}{a}$
Slope of **L'** $= \frac{b}{n}$

$$\text{Parallel if } \frac{m}{a} = \frac{b}{n}$$

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix}$$

$$mn = ab \Leftrightarrow mn - ab = 0 \Leftrightarrow det(A) = 0$$

Characteristic Equation: $\lambda^2 + (m+n)\lambda + (mn - ab) = 0$

$$\lambda^2 + (m+n)\lambda = 0$$

$$\lambda(\lambda + (m+n)) = 0 \Rightarrow \lambda = 0, \lambda = -(m+n)$$

ZERO AS AN EIGENVALUE EXAMPLE

$$m = 3, a = 6, n = 8, b = 4$$

$$A = \begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix}$$

$$\det \begin{bmatrix} -3-\lambda & 6 \\ 4 & -8-\lambda \end{bmatrix}$$

$$= (-3-\lambda)(-8-\lambda) - 24 = \lambda^2 + 11\lambda + 24 - 24$$

$$= \lambda^2 + 11\lambda = \lambda(\lambda + 11)$$

$$\lambda = 0, \lambda = -11$$

For $\lambda = -11$:

$$\begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4v_1 + 3v_2 = 0$$

$$\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

For $\lambda = 0$:

$$\begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3w_1 + 6w_2 = 0$$

$$\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = -11, \vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\lambda = 0, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

General Solution To

$$x' = -3x + 6y$$

$$y' = 4x = 8y$$

is $\mathbf{x} = C_1 e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-11t} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$x = 2C_1 - 3C_2 e^{-11t}$$

$$y = C_1 + 4C_2 e^{-11t}$$

Particular Solution: (x_0, y_0) at $t = 0$

$$x_0 = 2C_1 - 3C_2$$
$$y_0 = C_1 + 4C_2$$

$2C_1 - 3C_2 = x_0$	$8C_1 - 12C_2 = 4x_0$
$-2C_1 - 8C_2 = -2y_0$	$3C_1 + 12C_2 = 3y_0$
Add Equations	Add Equations
$-11C_2 = x_0 - 2y_0$	$11C_1 = 4x_0 + 3y_0$

$$C_1 = \frac{4x_0 + 3y_0}{11}, C_2 = \frac{-x_0 + 2y_0}{11}$$

$$x = 2(\frac{4x_0 + 3y_0}{11}) - 3(\frac{-x_0 + 2y_0}{11})e^{-11t}$$

$$y = \frac{4x_0 + 3y_0}{11} + 4(\frac{-x_0 + 2y_0}{11})e^{-11t}$$

Next Time

Complex Eigenvalues