

# MATH 226 Differential Equations

**EIGENVALUES**

**DEFINITION:**  
A SQUARE MATRIX  $A$  HAS **EIGENVALUE**  $\lambda$  IF & ONLY IF

$$A\mathbf{v} = \lambda\mathbf{v}$$

FOR SOME **NONZERO**  $\mathbf{v}$  WHICH IS AN **EIGENVECTOR**

**COMPUTATION:**  
TO FIND ALL THE EIGENVALUES  $\lambda$  SOLVE THE EQUATION

$$|(A - \lambda I)| = 0$$

October 7, 2022



Assignment 8  
Eigenvalues in MATLAB  
Reviewing For Exam 1

# Announcements

- ▶ First Team Projects Due Today By 5 PM
- ▶ Exam 1
  - ▶ **WEDNESDAY**
  - ▶ 7 PM - ? (No Time Limit)
  - ▶ 101 Warner
  - ▶ No Calculators, Books, Notes, Smart Phones, etc.
  - ▶ Focus on Material in Chapters 1 and 2

# Today's Topics

## Introduction To Systems of First Order Differential Equations

- ▶ Lotka – Volterra Predator Prey Model
- ▶ Richardson Arms Race Model
- ▶ Kermack – McKendrick Epidemic Model
- ▶ Home Heating Model
- ▶ Terrorism Recruitment Model

Initial Concern: **Homogeneous Systems of 2 First Order Differential Equations With Constant Coefficients**

$$x' = ax + by, y' = cx + dy$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

**Ideas and Tools From Linear Algebra Are Essential To This Study**

## EIGENVALUES AND EIGENVECTORS

$A$   $n \times n$  [Square Matrix]

$\vec{x}$   $n \times 1$  [Element of  $R^n$ ]

Then  $A\vec{x}$  is another vector in  $R^n$ .

Is there a **nonzero** vector  $\vec{v}$  and a constant  $\lambda$  such that

$$A\vec{v} = \lambda\vec{v}?$$

The equation  $A\vec{v} = \lambda\vec{v}$  is equivalent to

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$\text{or } (A - \lambda I)\vec{v} = \vec{0}$$

which is a system of homogeneous equations.

**The system has nontrivial solution if and only if**

**$(A - \lambda I)$  is Non-Invertible.**

$$\Rightarrow \det(A - \lambda I) = 0.$$

## Example

$$x' = -13x + 6y$$

$$y' = 2x - 2y$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{x}' = A\vec{x} \text{ or } \mathbf{X}' = A \mathbf{X}$$

Looks like  $x' = ax$  which has solution  $x = Ce^{at}$ .

Could there be a scalar  $\lambda$  and **nonzero** vector  $\vec{v}$  such that

$$\vec{x} = e^{\lambda t} \vec{v} \text{ is a solution?}$$

$$\begin{aligned} \vec{x}' = A\vec{x} \text{ becomes } \lambda e^{\lambda t} \vec{v} &= A e^{\lambda t} \vec{v} \\ \text{or } A\vec{v} = \lambda \vec{v} &\Rightarrow (A - \lambda I)\vec{v} = \vec{0}. \end{aligned}$$

## Finding Eigenvalues and Associated Eigenvectors

$$\text{Example : } A = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -13 - \lambda & 6 \\ 2 & -2 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-13 - \lambda)(-2 - \lambda) - (2)(6)$$

$$\det(A - \lambda I) = 26 + 13\lambda + 2\lambda + \lambda^2 - 12$$

$$\det(A - \lambda I) = \lambda^2 + 15\lambda + 14 = (\lambda + 14)(\lambda + 1)$$

$$\lambda = -14 \text{ or } \lambda = -1.$$



$$A = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix}$$

$$\text{For } \lambda = -1, A - \lambda I = \begin{bmatrix} -13 - (-1) & 6 \\ 2 & -2 - (-1) \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -12 & 6 \\ 2 & -1 \end{bmatrix}$$

$$\text{Row Reduces to } \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-2v_1 + v_2 = 0 \text{ so } v_2 = 2v_1$$

so a corresponding eigenvector is  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

One Solution to  $\vec{x}' = A\vec{x}$  is  $e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix}$$

$$\text{For } \lambda = -14, A - \lambda I = \begin{bmatrix} -13 - (-14) & 6 \\ 2 & -2 - (-14) \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 6 \\ 2 & 12 \end{bmatrix}$$

$$\text{Row Reduces to } \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix}$$

$$w_1 + 6w_2 = 0 \text{ so } w_2 = -\frac{1}{6}w_1$$

so a corresponding eigenvector is  $\begin{bmatrix} 6 \\ -1 \end{bmatrix}$ .

Another Solution to  $\vec{x}' = A\vec{x}$  is  $e^{-14t} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

Suppose  $X_1$  and  $X_2$  are each solutions of  $\mathbf{X}' = \mathbf{A}\mathbf{X}$  .

Let  $\alpha$  and  $\beta$  be any two constants.

Claim  $\alpha X_1 + \beta X_2$  is also a solution

Proof: On One Hand:

$$(\alpha X_1 + \beta X_2)' = \alpha X_1' + \beta X_2' = \alpha A X_1 + \beta A X_2$$

On Other Hand :

$$A(\alpha X_1 + \beta X_2) = \alpha A X_1 + \beta A X_2$$

**The set of solutions to  $\mathbf{X}' = \mathbf{A}\mathbf{X}$  is a  
VECTOR SPACE.**

**Theorem:** Suppose  $\lambda \neq \mu$  are two distinct eigenvalues of a square matrix  $A$  with respective eigenvectors  $\vec{v}$  and  $\vec{w}$ ; That is,

$$A\vec{v} = \lambda\vec{v} \text{ and } A\vec{w} = \mu\vec{w}$$

Then  $\{\vec{v}, \vec{w}\}$  is a Linearly Independent set of vectors.

Proof: Suppose  $a$  and  $b$  are constants such that

$$(*) \quad a\vec{v} + b\vec{w} = \vec{0}$$

First, Multiply (\*) by  $A$ :

$$aA\vec{v} + bA\vec{w} = A\vec{0} = \vec{0}$$

$$(**) \quad a\lambda\vec{v} + b\mu\vec{w} = \vec{0}$$

Next, Multiply (\*) by  $\mu$  to obtain

$$(***) \quad a\mu\vec{v} + b\mu\vec{w} = \vec{0}$$

Now subtract (\*\*\*) from (\*\*):

$$a(\lambda - \mu)\vec{v} = \vec{0}$$

Since  $\lambda \neq \mu$  and  $\vec{v} \neq \vec{0}$ , we must have  $a = 0$ .

But this means  $b\vec{w} = \vec{0}$  and hence  $b = 0$ .