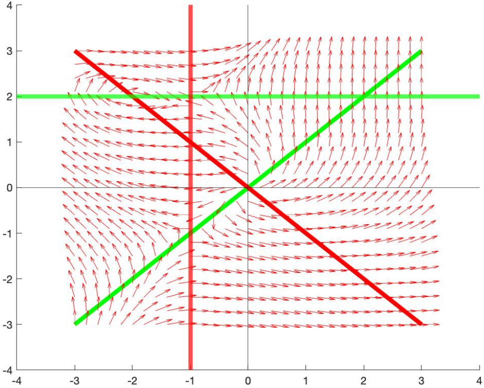
MATH 226: Differential Equations



Class 26: November 11, 2022

Assignment 17
MATLAB Autonomous Example

Exam 2

Wednesday, November 16

7 PM – ?
Here

General System of First Order Differential Equations

$$x_1' = f_1(x_1, x_2, ..., x_n, t)$$

$$x_2' = f_2(x_1, x_2, ..., x_n, t)$$

$$\vdots$$

$$x_n' = f_n(x_1, x_2, ..., x_n, t)$$

$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

Autonomous Systems

No Explicit t on Right Hand Side

$n = 2$	$n = 3$
$x' = F(x, y)$	$x' = F(x, y, z)$
$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$

Three Approaches:

Analytic: Rarely Possible To Find Closed Form Solution

Numeric: Detailed Information About a Single Solution

Geometric: Qualitative Information About All Solutions

Special Properties of Autonomous Systems

1. Direction Field is Independent of Time
2. Only One Trajectory Passing Through Each Point (x_0, y_0)
3. A Trajectory Can Not Cross Itself
4. A Single Well-Chosen Phase Portrait Simultaneously Displays Important Information About All Solutions

Analyzing a Nonlinear System

An Example:

$$x' = F(x,y) = (2-y)(x-y) = 2x - 2y - xy + y^2$$
$$y' = G(x,y) = (1+x)(x+y) = x + y + x^2 + xy$$

Step 1: Find Critical Points: $x' = 0$ and $y' = 0$
Step 2: Find Signs of x' and y'
Step 3: Linearize Near Critical Points

Example

$$x' = (2 - y)(x - y)$$
$$y' = (1 + x)(x + y)$$

STEP ONE: Identify All Equilibrium Points

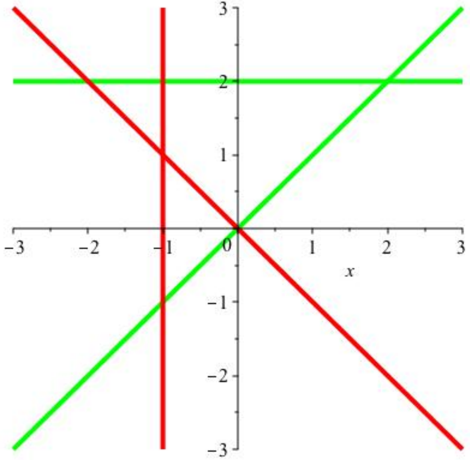
x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x

$$x' = (2-y)(x-y), y' = (1+x)(x+y)$$

x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x

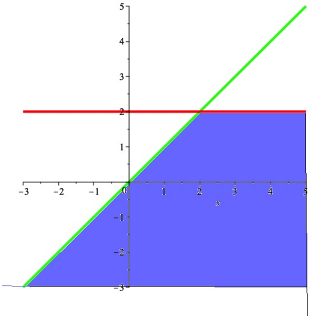


Step 2: Find Signs of x' and y'

$$x' = (2 - y)(x - y) > 0$$

CASE 1: BOTH FACTORS POSITIVE

$$2 - y > 0 \quad x - y > 0$$
$$2 > y \qquad x > y$$
$$y < 2 \qquad y < x$$

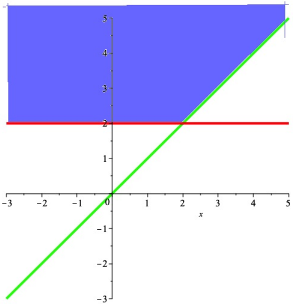


Step 2: Find Signs of x' and y'

$$x' = (2 - y)(x - y) > 0$$

CASE 2: BOTH FACTORS NEGATIVE

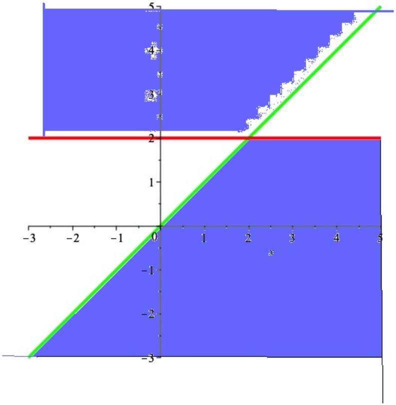
$$\begin{array}{ll} 2 - y < 0 & x - y < 0 \\ 2 < y & x < y \\ y > 2 & y > x \end{array}$$



Step 2: Find Signs of x' and y'

x' Positive

$$x' = (2-y)(x-y) > 0$$

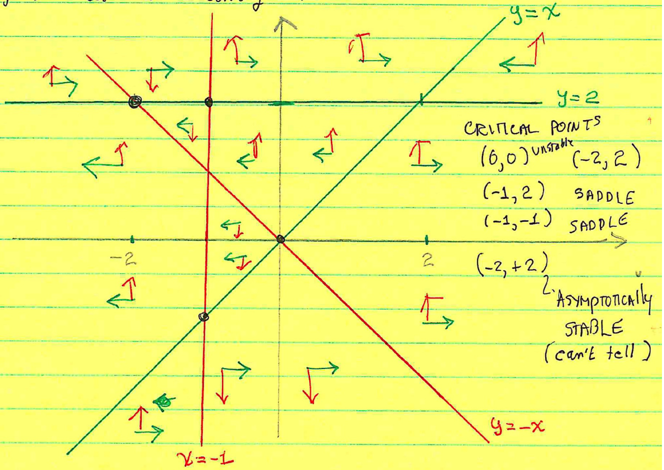


EXAMPLE $x' = (2-y)(x-y) = 2x - 2y - xy + y^2$

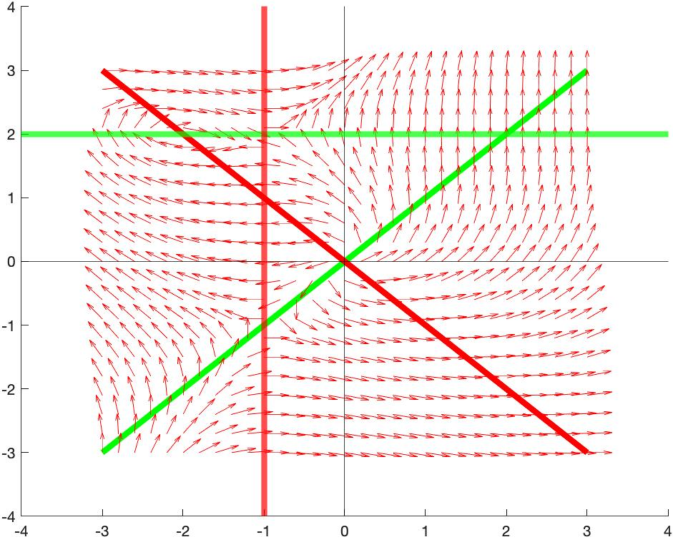
$y' = (1+x)(x+y) = x + y + x^2 + xy$

$x' = 0$ at $y = 2$ and $y = x$ GREEN

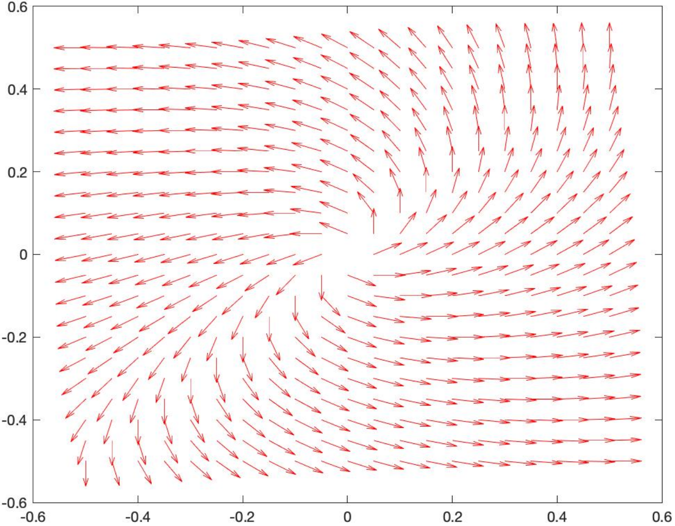
$y' = 0$ at $x = -1$ and $y = -x$



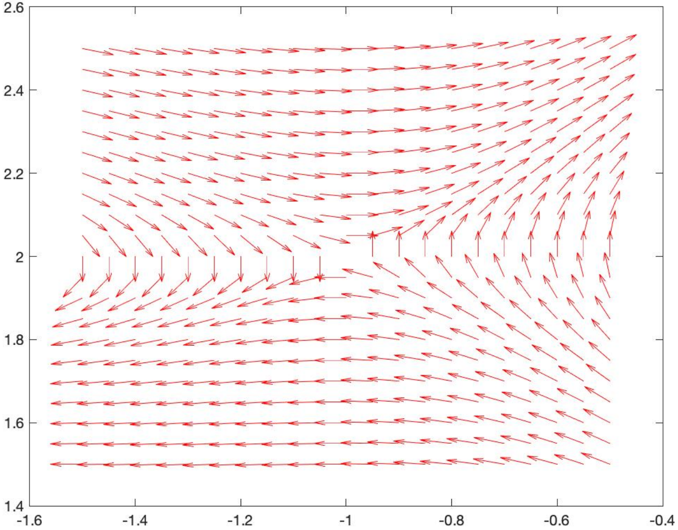
Examine Direction Field



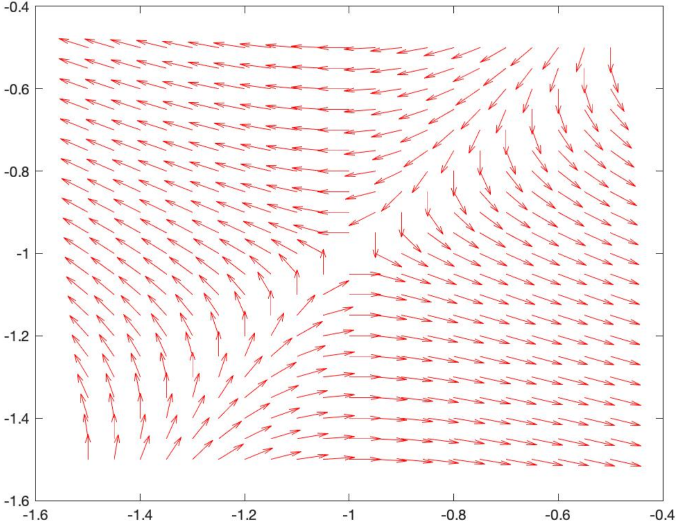
Examine Direction Field in Neighborhood of (0,0)



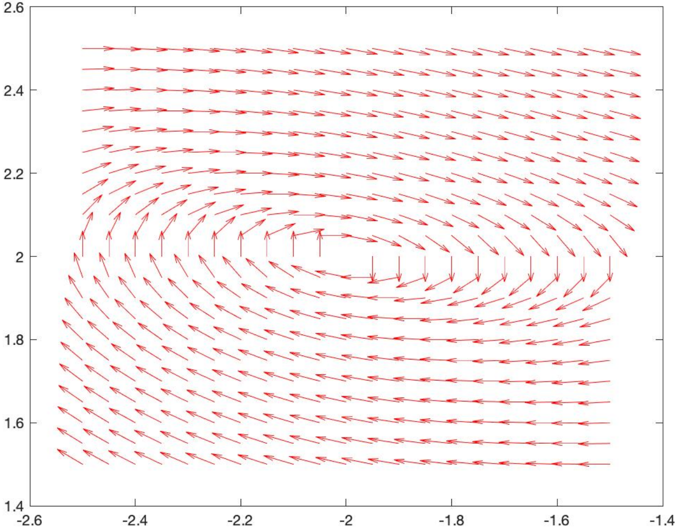
Examine Direction Field in Neighborhood of (-1,2)



Examine Direction Field in Neighborhood of (-1,-1)



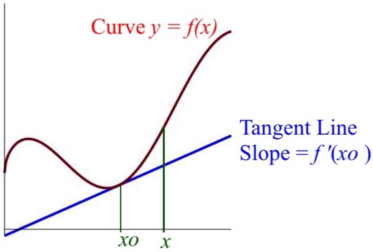
Examine Direction Field in Neighborhood of (-2,2)



Approximating a Function

Classic Calculus Example: $f : \mathbb{R}^1 \to \mathbb{R}^1$

Approximate Curve with a Tangent LIne



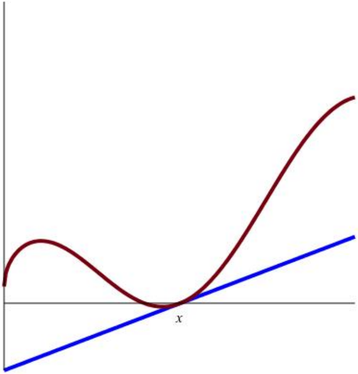
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0 + h) \approx f(x_0) + f'(x_0)(h)$$

To Do Better, Use Taylor Series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \ldots$$

Approximating a Function With Tangent Line
Special Case: $f(x_0) = 0$



$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = f'(x_0)(x - x_0)$$
$$f(x_0 + h) \approx f'(x_0)(h)$$

Approximating Function of Two Variables

$$F(x,y) \approx F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$$

$$F(x,y) \approx F(x_0,y_0) + (F(x_0,y_0), F_y(x_0,y_0)) \cdot (x-x_0, y-y_0)$$

$$F(x,y) \approx F(x_0,y_0) + \nabla F(x_0,y_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

Let $\mathbf{x} = (x,y), \mathbf{x}_0 = (x_0,y_0)$:

$$F(\mathbf{x}) \approx F(\mathbf{x}_0) + \nabla F(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

If $\mathbf{x}_0$ is a critical point, then $F(\mathbf{x}) \approx \nabla F(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$

$$f(x) \approx f'(x_0)(x-x_0)$$

$$\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$$

Near Critical Point, System Behaves Like **X'** $=$ A **X** where

$$A = \begin{bmatrix} F_x(x^*, y^*) & F_y(x^*, y^*) \\ G_x(x^*, y^*) & G_y(x^*, y^*) \end{bmatrix} = J(x^*, y^*)$$

Current Goal:

Approximating Nonlinear Autonomous System with Linear System Near An Equilibrium Point

$$x' = F(x,y)$$
$$y' = G(x,y)$$

$$F(x,y) \approx F(x^*,y^*) + F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G(x^*,y^*) + G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

But at Equilibrium Points, $F(x^*,y^*) = 0, G(x^*,y^*) = 0$ so

$$F(x,y) \approx F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

Carl Gustav Jacob Jacobi
December 10, 1804 – February 18, 1851

$$F(x,y) \approx F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

which we can write as

$$\begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix} \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix}$$

or

$$J(x^*,y^*) \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix} = J(x^*,y^*) \begin{bmatrix} h \\ k \end{bmatrix}$$

J is called the **Jacobi Matrix** or **Jacobian**

Example $x' = (2-y)(x-y) = 2x - 2y - xy + y^2 = F(x,y)$

$$y' = (1+x)(x+y) = x + y + x^2 + xy = G(x,y)$$

$$F_x = 2 - y \qquad G_x = 1 + 2x + y$$

$$F_y = -2 - x + 2y \qquad G_y = 1 + x$$

(x^*, y^*)	$J(x^*, y^*)$	Eigenvalues	Nature
(0,0)	$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$	$\frac{1}{2}(3 \pm i\sqrt{7})$	Unstable Spiral
(-1,-1)	$\begin{pmatrix} 3 & -3 \\ -2 & 0 \end{pmatrix}$	$\frac{1}{2}(3 \pm \sqrt{33})$	Saddle Point
(-1,2)	$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$	$\pm\sqrt{3}$	Saddle Point
(-2,2)	$\begin{pmatrix} 0 & 4 \\ -1 & -1 \end{pmatrix}$	$\frac{1}{2}(-1 \pm i\sqrt{15})$	Asymptotically Stable Spiral

Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x, y)$$
$$y' = my - nxy = y(m - nx) = G(x, y)$$
$$a, b, m, n > 0$$

Search For Equilibrium Points

$x' = 0$ at $x = 0, y = \frac{a}{b}$

$y' = 0$ at $y = 0, x = \frac{m}{n}$

Critical Points: (0,0) and $(\frac{m}{n}, \frac{a}{b})$: One Point Orbits

x-axis is an orbit and y-axis is an orbit.

$x' > 0$	$y' > 0$
$x(a - by) > 0$	$y(m - nx) > 0$
Both Positive	Both Positive
$x > 0$ and $y < \frac{a}{b}$	$y > 0$ and $x < \frac{m}{n}$
OR	OR
Both Negative	Both Negative
$x < 0$ and $y > \frac{a}{b}$	$y >=< 0$ and $x > \frac{m}{n}$

$x' > 0$	$y' > 0$
$x(a - by) > 0$	$y(m - nx) > 0$
Both Positive	Both Positive
$x > 0$ and $y < \frac{a}{b}$	$y > 0$ and $x < \frac{m}{n}$
OR	OR
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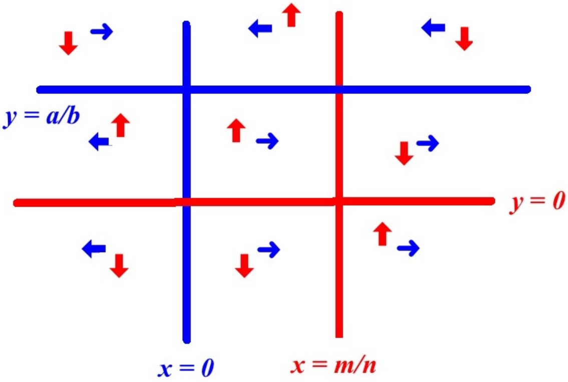
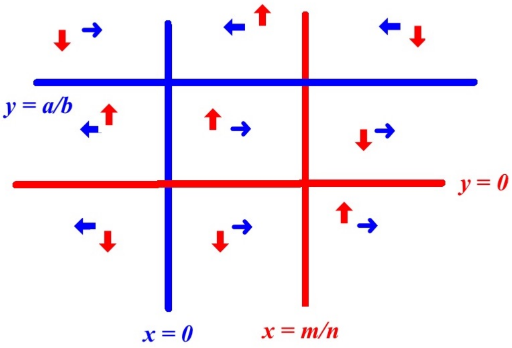
$x < 0$ and $y > \frac{a}{b}$	$y >=< 0$ and $x > \frac{m}{n}$



Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x, y)$$
$$y' = my - nxy = y(m - nx) = G(x, y)$$
$$a, b, m, n > 0$$
$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$
$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & m \end{bmatrix}$$

Both Eigenvalues are positive

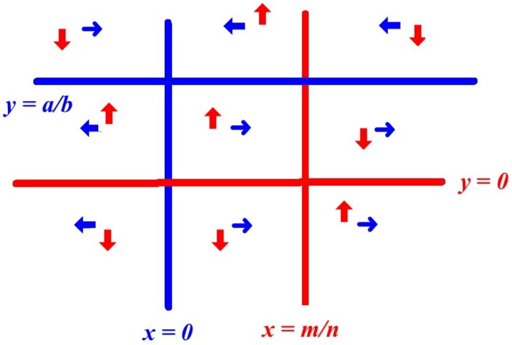


$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$
$$J(\tfrac{m}{n}, \tfrac{a}{b}) = \begin{bmatrix} a - b\frac{a}{b} & -b\frac{m}{n} \\ -n\frac{a}{b} & m - n\frac{m}{n} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{bm}{n} \\ -\frac{na}{b} & 0 \end{bmatrix}$$

Characteristic Polynomial $\lambda^2 - \frac{-an}{b}\frac{bm}{n} = \lambda^2 - am$

Eigenvalues $\pm\sqrt{am}$ so we have Saddle Point.

Eigenvectors: $\mathbf{v} = \begin{pmatrix} -bm \\ n\sqrt{am} \end{pmatrix}, \mathbf{w} = \begin{pmatrix} bm \\ n\sqrt{am} \end{pmatrix}$



A Specific Example: $a = .9, b = .6, m = .4, n = .3$

Critical Point $(\frac{.4}{.3}, \frac{.9}{.6}) = (\frac{4}{3}, \frac{3}{2})$

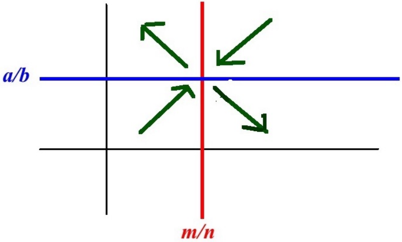
$$J() = \begin{bmatrix} 0 & -\frac{(.6)(.4)}{.3} \\ -\frac{.27}{.6} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{4}{5} \\ -\frac{9}{20} & 0 \end{bmatrix}$$

Eigenvalue	Eigenvector	Solution
$\lambda = \sqrt{36/100} = \frac{3}{5}$	$\mathbf{v} = \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$	$e^{\frac{3}{5}t}\mathbf{v}$
$\lambda = -\sqrt{(36/100} = -\frac{3}{5}$	$\mathbf{w} = \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$	$e^{-\frac{3}{5}t}\mathbf{w}$

General Solution of Linear System

$$C_1 e^{\frac{3}{5}t} \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} + C_2 e^{-\frac{3}{5}t} \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$$

Behavior Near Equilibrium Point



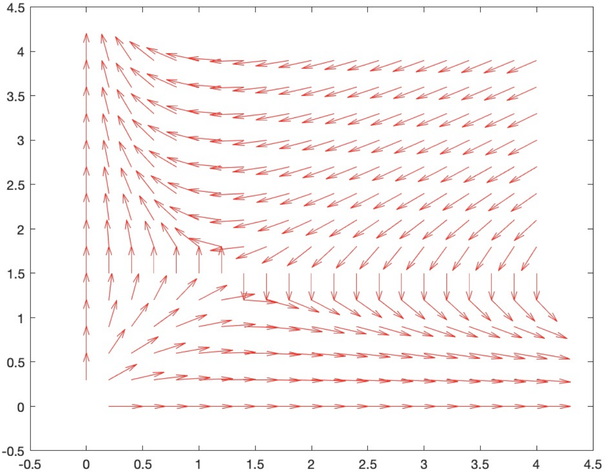
$$C_1 e^{\frac{3}{5}t}\begin{pmatrix}-4/3\\1\end{pmatrix}+C_2 e^{-\frac{3}{5}t}\begin{pmatrix}4/3\\1\end{pmatrix}$$

Simple Model Of Competition $x' = ax - bxy$

$$y' = my - nxy$$

$$a, b, m, n > 0$$

Each Species Grows **Exponentially** in Absence of Other.



Simple Model $x' = ax - bxy$

$$y' = my - nxy$$

$$a, b, m, n > 0$$

Each Species Grows **Exponentially** in Absence of Other.

More Realistic Model

$$x' = ax - px^2 - bxy$$

$$y' = my - qy^2 - nxy$$

$$a, b, m, n > 0$$

Each Spcies Grows **Logistically** in Absence of Other.

Next Time

Predator-Prey Models