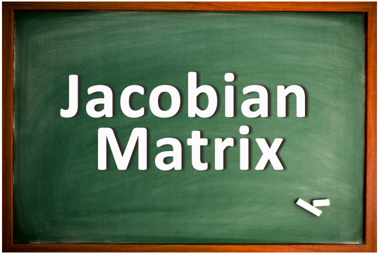
MATH 226: Differential Equations



Class 25: November 9, 2022

Assignment 15
MATLAB Simple Competition Model

Maple Examples in Handouts Folder:
Simple Competition Model
Competition Model

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University of Washington
Today 12:30 – 1:20 PM in 75 Shannon Street Room 224

Child Mortality Estimation in a Low- and Middle-Income Country Context

One target of the Sustainable Development Goals established in 2015 by the United Nations is to reduce both neonatal and under-5 mortality in all countries by the year 2030. Each target includes specific metrics that must be met to achieve this goal. As statisticians, how can we assist countries in meeting these targets, and monitoring their progress? One answer is through providing accurate and precise estimates of neonatal and under-5 mortality over time. In a low- and middle-income country context, this is not straightforward, as we must wrestle with complex survey data, interval-censored survival outcomes, data sparsity, known issues with Demographic and Health Surveys, and specific concerns related to producing official statistics for government organizations. In this talk, we'll discuss how and why each of these concerns arise and how we can use statistical methods to address them.

Exam 2

Wednesday, November 16

7 PM – ?
Here

General System of First Order Differential Equations

$$x_1^{'} = f_1(x_1, x_2, ..., x_n, t)$$

$$x_2^{'} = f_2(x_1, x_2, ..., x_n, t)$$

$$\vdots$$

$$x_n^{'} = f_n(x_1, x_2, ..., x_n, t)$$

$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

Autonomous Systems

No Explicit t on Right Hand Side

$n = 2$	$n = 3$
$x' = F(x, y)$	$x' = F(x, y, z)$
$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$

Three Approaches:

Analytic: Rarely Possible To Find Closed Form Solution

Numeric: Detailed Information About a Single Solution

Geometric: Qualitative Information About All Solutions

Special Properties of Autonomous Systems

1. Direction Field is Independent of Time
2. Only One Trajectory Passing Through Each Point (x_0, y_0)
3. A Trajectory Can Not Cross Itself
4. A Single Well-Chosen Phase Portrait Simultaneously Displays Important Information About All Solutions

Analyzing a Nonlinear System

An Example:

$$x' = F(x, y) = (2 - y)(x - y) = 2x - 2y - xy + y^2$$
$$y' = G(x, y) = (1 + x)(x + y) = x + y + x^2 + xy$$

Step 1: Find Critical Points: $x' = 0$ and $y' = 0$
Step 2: Find Signs of x' and y'
Step 3: Linearize Near Critical Points

Example

$$x' = (2 - y)(x - y)$$
$$y' = (1 + x)(x + y)$$

STEP ONE: Identify All Equilibrium Points

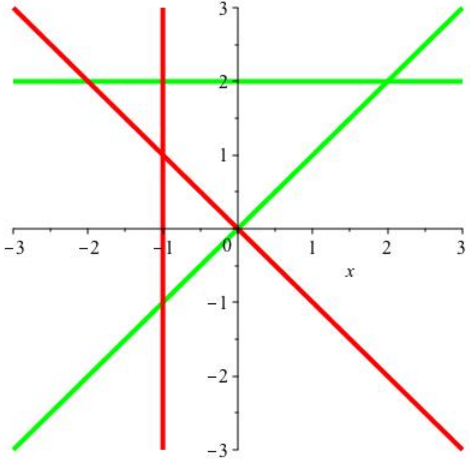
x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x

$$x' = (2-y)(x-y), y' = (1+x)(x+y)$$

x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x

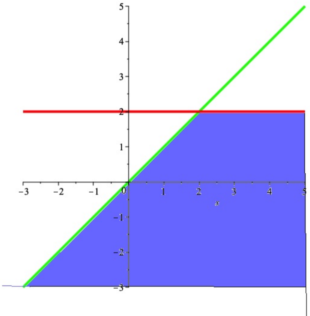


Step 2: Find Signs of x' and y'

$$x' = (2 - y)(x - y) > 0$$

CASE 1: BOTH FACTORS POSITIVE

$$\begin{array}{cc} 2 - y > 0 & x - y > 0 \\ 2 > y & x > y \\ y < 2 & y < x \end{array}$$

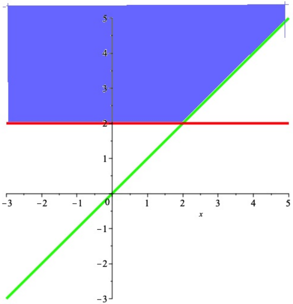


Step 2: Find Signs of x' and y'

$$x' = (2 - y)(x - y) > 0$$

CASE 2: BOTH FACTORS NEGATIVE

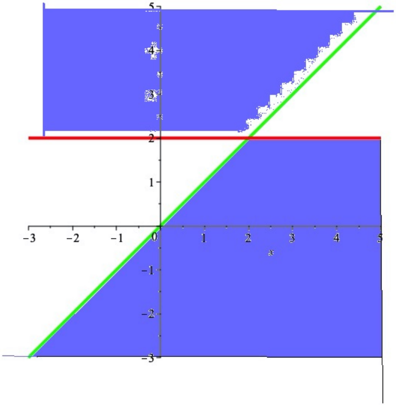
$$\begin{array}{ll} 2 - y < 0 & x - y < 0 \\ 2 < y & x < y \\ y > 2 & y > x \end{array}$$



Step 2: Find Signs of x' and y'

x' Positive

$$x' = (2 - y)(x - y) > 0$$

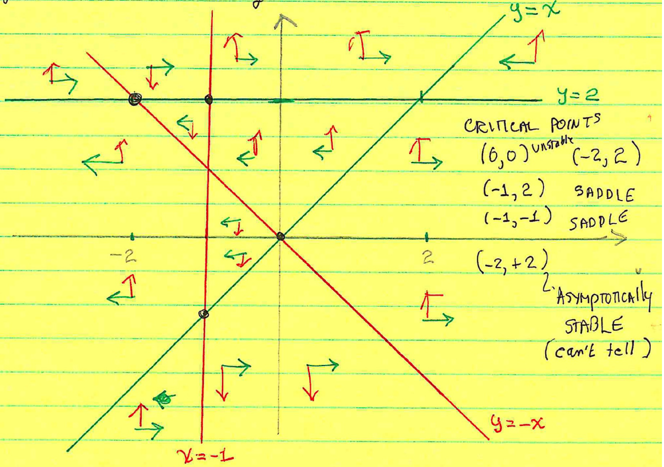


EXAMPLE $x' = (2-y)(x-y) = 2x - 2y - xy + y^2$

$y' = (1+x)(x+y) = x + y + x^2 + xy$

$x' = 0$ at $y = 2$ and $y = x$ GREEN

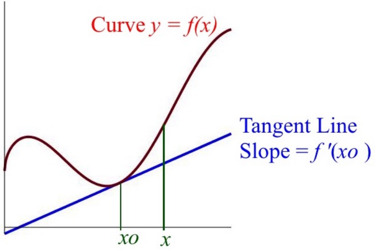
$y' = 0$ at $x = -1$ and $y = -x$



Approximating a Function

Classic Calculus Example: $f : \mathbb{R}^1 \to \mathbb{R}^1$

Approximate Curve with a Tangent LIne



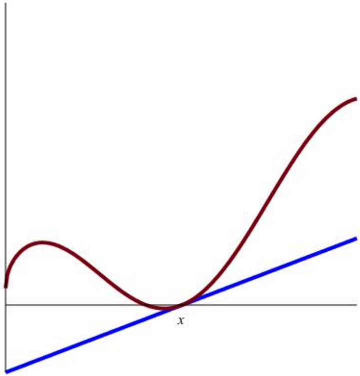
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$
$$f(x_0 + h) \approx f(x_0) + f'(x_0)(h)$$

To Do Better, Use Taylor Series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \ldots$$

Approximating a Function With Tangent Line

Special Case: $f(x_0) = 0$



$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = f'(x_0)(x - x_0)$$
$$f(x_0 + h) \approx f'(x_0)(h)$$

Approximating Function of Two Variables

$$F(x,y) \approx F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$$

$$F(x,y) \approx F(x_0,y_0) + (F(x_0,y_0), F_y(x_0,y_0)) \cdot (x-x_0, y-y_0)$$

$$F(x,y) \approx F(x_0,y_0) + \nabla F(x_0,y_0) \cdot (\mathbf{x}-\mathbf{x}_0)$$

Let $\mathbf{x} = (x,y), \mathbf{x}_0 = (x_0,y_0)$:

$$F(\mathbf{x}) \approx F(\mathbf{x}_0) + \nabla F(\mathbf{x}_0) \cdot (\mathbf{x}-\mathbf{x}_0)$$

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

If $\mathbf{x}_0$ is a critical point, then $F(\mathbf{x}) \approx \nabla F(\mathbf{x}_0) \cdot (\mathbf{x}-\mathbf{x}_0)$

$$f(x) \approx f'(x_0)(x-x_0)$$

$$\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$$

Near Critical Point, System Behaves Like $\mathbf{X'} = A\,\mathbf{X}$ where

$$A = \begin{bmatrix} F_x(x^*, y^*) & F_y(x^*, y^*) \\ G_x(x^*, y^*) & G_y(x^*, y^*) \end{bmatrix} = J(x^*, y^*)$$

Current Goal:
Approximating Nonlinear Autonomous System with Linear System Near An Equilibrium Point

$$x' = F(x, y)$$
$$y' = G(x, y)$$

$$F(x, y) \approx F(x^*, y^*) + F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G(x^*, y^*) + G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

But at Equilibrium Points, $F(x^*, y^*) = 0, G(x^*, y^*) = 0$ so

$$F(x, y) \approx F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

Carl Gustav Jacob Jacobi
December 10, 1804 – February 18, 1851

$$F(x,y) \approx F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

which we can write as

$$\begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix} \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix}$$

or

$$J(x^*,y^*) \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix} = J(x^*,y^*) \begin{bmatrix} h \\ k \end{bmatrix}$$

J is called the **Jacobi Matrix** or **Jacobian**

Example $x' = (2-y)(x-y) = 2x - 2y - xy + y^2 = F(x,y)$

$$y' = (1+x)(x+y) = x + y + x^2 + xy = G(x,y)$$

$$F_x = 2 - y \qquad G_x = 1 + 2x + y$$

$$F_y = -2 - x + 2y \qquad G_y = 1 + x$$

(x^*, y^*)	$J(x^*, y^*)$	Eigenvalues	Nature
(0,0)	$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$	$\frac{1}{2}(3 \pm i\sqrt{7})$	Unstable Spiral
(-1,-1)	$\begin{pmatrix} 3 & -3 \\ -2 & 0 \end{pmatrix}$	$\frac{1}{2}(3 \pm \sqrt{33})$	Saddle Point
(-1,2)	$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$	$\pm\sqrt{3}$	Saddle Point
(-2,2)	$\begin{pmatrix} 0 & 4 \\ -1 & -1 \end{pmatrix}$	$\frac{1}{2}(-1 \pm i\sqrt{15})$	Asymptotically Stable Spiral

Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x, y)$$
$$y' = my - nxy = y(m - nx) = G(x, y)$$
$$a, b, m, n > 0$$

Search For Equilibrium Points

$x' = 0$ at $x = 0, y = \frac{a}{b}$

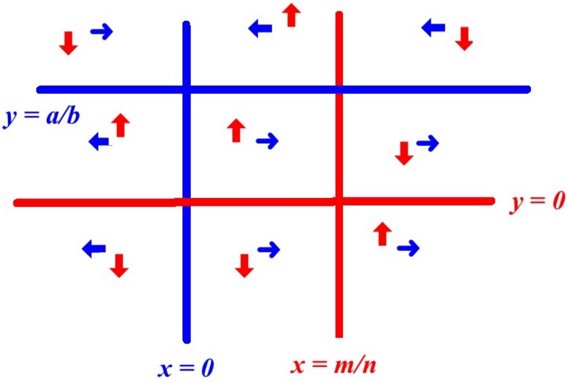
$y' = 0$ at $y = 0, x = \frac{m}{n}$

Critical Points: (0,0) and $(\frac{m}{n}, \frac{a}{b})$: One Point Orbits

x-axis is an orbit and y-axis is an orbit.

$x' > 0$	$y' > 0$
$x(a - by) > 0$	$y(m - nx) > 0$
Both Positive	Both Positive
$x > 0$ and $y < \frac{a}{b}$	$y > 0$ and $x < \frac{m}{n}$
OR	OR
Both Negative	Both Negative
$x < 0$ and $y > \frac{a}{b}$	$y >=< 0$ and $x > \frac{m}{n}$

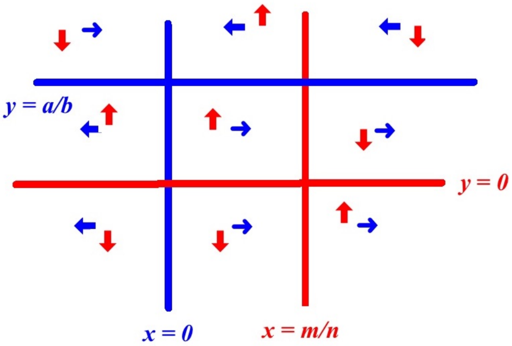
$x' > 0$	$y' > 0$
$x(a - by) > 0$	$y(m - nx) > 0$
Both Positive	Both Positive
$x > 0$ and $y < \frac{a}{b}$	$y > 0$ and $x < \frac{m}{n}$
OR	OR
Both Negative	Both Negative
$x < 0$ and $y > \frac{a}{b}$	$y >=< 0$ and $x > \frac{m}{n}$



Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x, y)$$
$$y' = my - nxy = y(m - nx) = G(x, y)$$
$$a, b, m, n > 0$$
$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$
$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & m \end{bmatrix}$$

Both Eigenvalues are positive

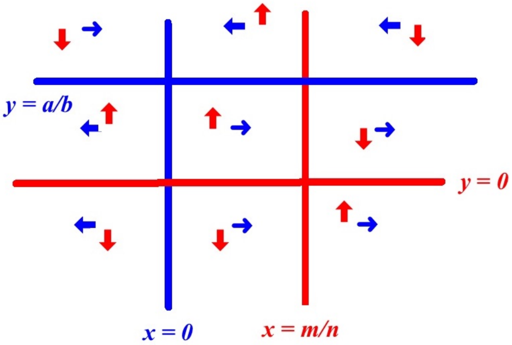


$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$
$$J\left(\frac{m}{n}, \frac{a}{b}\right) = \begin{bmatrix} a - b\frac{a}{b} & -b\frac{m}{n} \\ -n\frac{a}{b} & m - n\frac{m}{n} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{bm}{n} \\ -\frac{na}{b} & 0 \end{bmatrix}$$

Characteristic Polynomial $\lambda^2 - \frac{-an}{b}\frac{bm}{n} = \lambda^2 - am$

Eigenvalues $\pm\sqrt{am}$ so we have Saddle Point.

Eigenvectors: $\mathbf{v} = \begin{pmatrix} -bm \\ n\sqrt{am} \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} bm \\ n\sqrt{am} \end{pmatrix}$



A Specific Example: $a = .9, b = .6, m = .4, n = .3$

Critical Point $\left(\frac{.4}{.3}, \frac{.9}{.6}\right) = \left(\frac{4}{3}, \frac{3}{2}\right)$

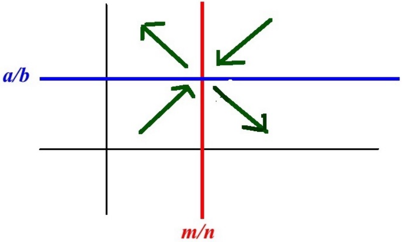
$$J() = \begin{bmatrix} 0 & -\frac{(.6)(.4)}{.3} \\ -\frac{.27}{.6} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{4}{5} \\ -\frac{9}{20} & 0 \end{bmatrix}$$

Eigenvalue	Eigenvector	Solution
$\lambda = \sqrt{36/100} = \frac{3}{5}$	$\mathbf{v} = \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$	$e^{\frac{3}{5}t}\mathbf{v}$
$\lambda = -\sqrt{(36/100} = -\frac{3}{5}$	$\mathbf{w} = \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$	$e^{-\frac{3}{5}t}\mathbf{w}$

General Solution of Linear System

$$C_1 e^{\frac{3}{5}t} \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} + C_2 e^{-\frac{3}{5}t} \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$$

Behavior Near Equilibrium Point



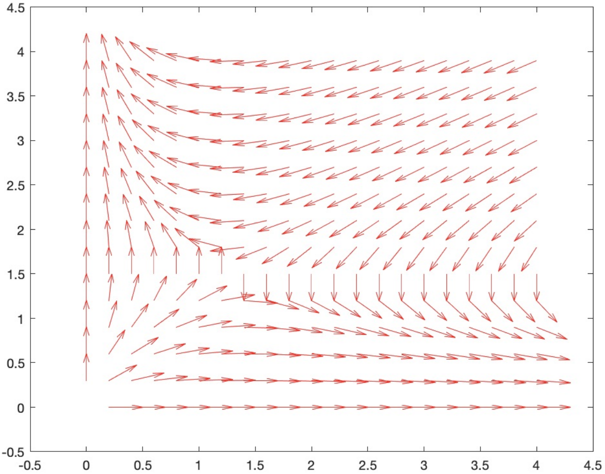
$$C_1 e^{\frac{3}{5}t} \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} + C_2 e^{-\frac{3}{5}t} \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$$

Simple Model Of Competition $x' = ax - bxy$

$$y' = my - nxy$$

$$a, b, m, n > 0$$

Each Species Grows **Exponentially** in Absence of Other.



Simple Model $x' = ax - bxy$

$$y' = my - nxy$$

$$a, b, m, n > 0$$

Each Species Grows **Exponentially** in Absence of Other.

More Realistic Model

$$x' = ax - px^2 - bxy$$

$$y' = my - qy^2 - nxy$$

$$a, b, m, n > 0$$

Each Spcies Grows **Logistically** in Absence of Other.

Next Time

Predator-Prey Models