A Unified Treatment of Tangent Lines and Tangent Planes

Tangent Lines to Curves

Case I: $f:\mathbb{R}^1 \to \mathbb{R}^1$ (Coordinate representation of the curve)

Here the **graph** of *f* is the curve and *y = f(x)* is equation of the curve

The tangent line to the graph of f at x = a is tangent line at (*a, f(a)*) and has equation $y = f(a) + f'(a)(x-a)$

Case 2: $f:\mathbb{R}^1 \to \mathbb{R}^2$ (Parametric Representation of the curve)

Here the ***image*** of *f* is the curve

We focus on the tangent line to the image of *f*.

Here $\mathbf{f}(t) = (f_1(t), f_2(t))$ and $\mathbf{f}'(t) = (f_1'(t), f_2'(t))$

Then: Tangent line is set of vectors of the form $\mathbf{f}(a) + \mathbf{f}'(a)t$

Case 3: $f:\mathbb{R}^1 \to \mathbb{R}^m$

Here the ***image*** of *f* is the curve

Now $\mathbf{f}(t) = (f_1(t), f_2(t), \ldots f_m(t))$ and $\mathbf{f}'(t) = (f_1'(t), f_2'(t), \ldots, f_m'(t))$

Again, Tangent line is set of vectors of the form $\mathbf{f}(a) + \mathbf{f}'(a)t$

Tangent Planes to Surfaces

Case 1: $f:\mathbb{R}^2 \to \mathbb{R}^1$ (coordinate representation of the curve)

Here the ***graph*** of *f* is the surface

Now *z = f(x,y)* is the equation of the surface.

If *(a,b)* is a point in $\mathbb{R}^2$ in the domain of *f*, then the tangent plane to the surface at *(a,b,f(a,b))* is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Note that we can write this equation in a number of equivalent ways:

$$z = f(a,b) + \left(f_x(a,b), f_y(a,b)\right) \bullet (x-a, y-b)$$

$$z = f(a,b) + \left(f_x(a,b), f_y(a,b)\right) \bullet \left((x,y)-(a,b)\right)$$

$$z = f(\mathbf{a}) + f'(\mathbf{a}) \bullet (\mathbf{x}-\mathbf{a}) \text{ where } \mathbf{a} = (a,b),\ \mathbf{x} = (x,y), \text{ and } f' = (f_x, f_y) = \nabla f$$

Case 2: $f:\mathbb{R}^2 \to \mathbb{R}^3$ (parametric representation of the surface)

Here the **image** of *f* is the surface

Now $f(u,v) = \left(f_1(u,v), f_2(u,v), f_3(u,v)\right)$

In vector form, with **a** = *(u,v),* and $f(\mathbf{a}) = \left(f_1(\mathbf{a}), f_2(\mathbf{a}), f_3(\mathbf{a})\right)$

Let $\dfrac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{pmatrix}$, $\dfrac{\partial f}{\partial v} = \begin{pmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial v} \end{pmatrix}$, and $f' = \nabla f = \left(\dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial v}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}$

The tangent plane to surface at $f(\mathbf{a})$ is the set of vectors of the form $f(\mathbf{a}) + f'(\mathbf{a})\begin{pmatrix} u \\ v \end{pmatrix}$

We can write out this vector formula as $\begin{pmatrix} f_1(\mathbf{a}) \\ f_2(\mathbf{a}) \\ f_3(\mathbf{a}) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial u}(\mathbf{a}) & \frac{\partial f_1}{\partial v}(\mathbf{a}) \\ \frac{\partial f_2}{\partial u}(\mathbf{a}) & \frac{\partial f_2}{\partial v}(\mathbf{a}) \\ \frac{\partial f_3}{\partial u}(\mathbf{a}) & \frac{\partial f_3}{\partial v}(\mathbf{a}) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{a}) + \frac{\partial f_1}{\partial u}(\mathbf{a})u + \frac{\partial f_1}{\partial v}(\mathbf{a})v \\ f_2(\mathbf{a}) + \frac{\partial f_2}{\partial u}(\mathbf{a})u + \frac{\partial f_2}{\partial v}(\mathbf{a})v \\ f_3(\mathbf{a}) + \frac{\partial f_3}{\partial u}(\mathbf{a})u + \frac{\partial f_3}{\partial v}(\mathbf{a})v \end{pmatrix}$

[We may solve the first two equations for *u* and *v* in terms of *x* and *y*. Then substitute into third equation to get *z* in terms of *x* and *y*.]

Summary: Equations for the Tangent Objects

	Curve	**Surface**
Coordinate Form	$y = f(a) + f'(a)(x-a)$	$z = f(\mathbf{a}) + f'(\mathbf{a}) \bullet (\mathbf{x} - \mathbf{a})$
Parametric Form	$\mathbf{f}(a) + \mathbf{f}'(a)t$	$f(\mathbf{a}) + f'(\mathbf{a})\begin{pmatrix} u \\ v \end{pmatrix}$
	Tangent is a Line	**Tangent is a Plane**