MATH 223

Some Notes on Assignment 3

Chapter 2: Exercises 21, 23, 26, 30, 31

21: For each of the following functions from $\mathbb{R}^1$ into $\mathbb{R}^m$, describe the set of points where the functions are defined and continuous:

1. $\mathbf{f}(t) = \left(t, \frac{1}{t}\right)$
   Solution: $f(t)$ is defined and continuous wherever t does not equal 0.
2. $\mathbf{f}(t) = \left(t^2, \sqrt{t}, \cos t, \tan t\right)$
   Solution: $f(t)$ is defined and continuous where $t \geq 0$ and $t \neq \frac{\pi}{2} + \pi n$ where n is a positive integer.
3. $\mathbf{f}(t) = \left(\ln t, \arctan t, \frac{t^2-1}{t-1}\right)$
   Solution: $f(t)$ is defined and continuous on the set $t > 0$ and $t \neq 1$.
4. $\mathbf{f}(t) = (t, t^2, t^3, \sin t, e^t)$
   Solution: $f(t)$ is defined and continuous on all real numbers.

23: Let $\mathcal{V}$ be the set of all functions from $\mathbb{R}^1$ into $\mathbb{R}^m$ which are continuous at the point a. Show that $\mathcal{V}$ is a vector space. Does $\mathcal{V}$ have finite dimension?
Solution: i) We already know that vector addition is commutative, associative and preserves continuity. Then, for all $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ included in $\mathcal{V}$

$$\mathbf{f} + \mathbf{g} = \mathbf{g} + \mathbf{f} \text{ and } (\mathbf{f} + \mathbf{g}) + \mathbf{h} = \mathbf{f} + (\mathbf{g} + \mathbf{h})$$

ii) The vector space $\mathcal{V}$ contains the zero vector $\mathbf{z}(t) = \mathbf{0}$.
iii) Every continuous function $\mathbf{f}$ has an additive inverse **-f** which is also continuous.
iv) The scalar 1 is the multiplicative identity for all continuous functions
v) The vector space $\mathcal{V}$ is infinitely dimensional. For each n, the set of vectors $\{x, x^2, ..., x^n\}$. is a linearly independent set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$, because no polynomial can be written as a linear combination of the others. Since we have a vector of the form x^n for every positive integer n, our vector space contains an infinitely large linearly independent set and thus has infinite dimension. [We are writing x^n as shorthand for the function $f(x) = x^n$].

26:. For each of the following vector-valued functions $\mathbf{f}$, find $\mathbf{f}'$

1. $\mathbf{f}(t) = (\cos 2t, \sin 2t)$
   Solution: $\mathbf{f}'(t) = (-2\sin 2t, 2\cos 2t)$
2. $\mathbf{f}(t) = (t^{3/2}, t^{5/2})$
   Solution: $\mathbf{f}'(t) = (\frac{3}{2}t^{1/2}, \frac{5}{2}t^{3/2})$

3. $\mathbf{f}(t) = (\sin 3t, e^{-4t}, \sqrt{t+1})$
   Solution: $\mathbf{f}'(t) = (3cos(3t), -4e^{-4t}, \frac{1}{2}(t+1)^{-1/2})$

4. $\mathbf{f}(t) = (t^2, t^3, t^4)$
   Solution: $\mathbf{f}'(t) = (2t, 3t^2, 4t^3)$

30: Prove Theorem 2.3.2 (c)
Solution:

$$(\mathbf{p} \cdot \mathbf{q})' = \text{ (by definition of dot product)}$$
$$(p_1q_1 + p_2q_2 + p_3q_3)' = \text{ (Derivative of sum is sum of derivatives)}$$
$$((p_1q_1)' + (p_2q_2)' + (p_3q_3)') = \text{ (by Product Rule)}$$
$$(p_1'q_1 + p_1q_1' + p_2'q_2 + p_2q_2' + p_3'q_3 + p_3q_3') = \text{ (rearrange terms}$$
$$(p_1'q_1 + p_2'q_2 + p_3'q_3) + (p_1q_1' + p_2q_2' + p_3q_3') = (\mathbf{p}' \cdot \mathbf{q}) + (\mathbf{p} \cdot \mathbf{q}')$$

31: Show that Theorem 2.3.2 implies $(\mathbf{p} - \mathbf{q})' = \mathbf{p}' - \mathbf{q}'$
Solution: By Theorem 2.3.2 we know that $(\mathbf{p} + \mathbf{g}) = \mathbf{p}' + \mathbf{g}'$ and $(\alpha\mathbf{q})' = \alpha\mathbf{q}'$
If we let $\alpha = -1$ and substitute $-\mathbf{q}$ for $\mathbf{g}$ we have

$$(\mathbf{p} + (-\mathbf{q}))' = \mathbf{p}' + (-\mathbf{q}')$$
$$(\mathbf{p} - \mathbf{q})' = \mathbf{p}' - \mathbf{q}'$$