MATH 223
Some Notes on Assignment 2
Chapter 2: Exercises 1, 4, 9, 10, 12, 13, 14

1. Find an equation of the ellipse with vertices at (4, 2) and (-6, 2) with an eccentricity of 4/5 .
Solution: The center of the ellipse is halfway between (-6,2) and (4,2) along the horizontal line $y = 2$; that is, (-1,2). Thus the equation has the form

$$\frac{(x+1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

for some values of a and b. Since (4,2) lies on the ellipse, the coordinates satisfy the equation. Thus

$$\frac{(4+1)^2}{a^2} + \frac{(2-2)^2}{b^2} = \frac{5^2}{a^2} = 1 \text{ so } a^2 = 5^2 = 25$$

Now we also given that the eccentricity is 4/5 so

$$\frac{\sqrt{a^2 - b^2}}{a} = \frac{4}{5}$$

which gives $b^2 = 9$. An equation for the ellipse is

$$\frac{(x+1)^2}{5^2} + \frac{(y-2)^2}{3^2} = 1$$

4.The orbits of planets around the sun are approximately elliptical with the sun as a focus. The *aphelion* is a planet's greatest distance from the sun and the *perihelion* is its least distance. The length of the major axis is the sum of the aphelion and the perihelion. Earth's aphelion is 94.51 million miles and its perihelion is 91.40 million miles. Write an equation for Earth's orbit.
Solution: Set up a coordinate system with the center of the ellipse at the origin.

The major axis is 94.51 + 91.40 = 185.91 so the vertices are at ±92.96 and the Sun's position is at (92.96 - 91.40,0) = (1.56,0) = (c,0). Thus $a^2 = 92.96^2$ and $b^2 = a^2 - c^2 = 92.96^2 - 1.56^2 = 8639.13 = 92.95^2$. An equation for the Earth's orbit is

$$\frac{x^2}{92.96^2} + \frac{y^2}{92.95^2} = 1$$

9.If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, show that the midpoint of the segment connecting them is $\frac{\mathbf{x}+\mathbf{y}}{2}$

Solution: We'll show that the distance between $\mathbf{x}$ and $\frac{\mathbf{x}+\mathbf{y}}{2}$ is the same as the distance between $\mathbf{y}$ and $\frac{\mathbf{x}+\mathbf{y}}{2}$.
Distance between $\mathbf{x}$ and $\frac{\mathbf{x}+\mathbf{y}}{2} = |\mathbf{x} - \frac{\mathbf{x}+\mathbf{y}}{2}| = |\frac{2\mathbf{x}-\mathbf{x}-\mathbf{y}}{2}| = |\frac{\mathbf{x}-\mathbf{y}}{2}| = \frac{|\mathbf{x}-\mathbf{y}|}{2}$.
Distance between $\mathbf{y}$ and $\frac{\mathbf{x}+\mathbf{y}}{2} = |\mathbf{y} - \frac{\mathbf{x}+\mathbf{y}}{2}| = |\frac{2\mathbf{y}-\mathbf{x}-\mathbf{y}}{2}| = |\frac{\mathbf{y}-\mathbf{x}}{2}| = \frac{|\mathbf{y}-\mathbf{x}|}{2}$
But $|\mathbf{x} - \mathbf{y}| = |\mathbf{y} - \mathbf{x}|$ so the two distances are equal.

10. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, show that the distance between them is given by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Solution: The distance between two vectors $\mathbf{x}$ and $\mathbf{y}$ is the magnitude of their difference $\mathbf{v} = \mathbf{x} - \mathbf{y}$. The magnitude of a vector is the square root of the sum of the squares of the individual components. Thus
$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = |\mathbf{v}| = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
since the ith component of $\mathbf{v}$ is $x_i - y_i$.

12: a) The vector $-\mathbf{x}$ is in the opposite direction of $\mathbf{x}$.
b)Any vector with same direction of $\mathbf{x}$ would be a positive scaled multiple of $\mathbf{x}$. Since we know the direction and magnitude of our desired vector, we can solve for the desired scalar a.

$$3 = \sqrt{(a2)^2 + (a(-3))^2 + (a)^2 + (a5)^2}$$

$$9 = a^2(4 + 9 + 1 + 25)$$

$$\frac{9}{39} = a^2$$

$$a = \frac{3}{\sqrt{3}}$$

Then a a vector having the same direction as $\mathbf{x}$ and magnitude 3 would be $\sqrt{\frac{3}{13}}\mathbf{x}$.

13. Show that the points (x, y) in the plane satisfying the equation

$$x^2 + y^2 - 8x - 10y + 32 = 0$$

form the circle of radius 3 with center (4,5).
Solution: Completing the square in x and y, we have
$x^2 - 8x = x^2 - 8x + 16 - 16 = (x - 4)^2 - 16$ and $y^2 - 10y = (y - 5)^2 - 25$ so
$x^2 + y^2 - 8x - 10y + 32 = (x-4)^2 - 16 + (y-5)^2 - 25 + 32 = (x-4)^2 + (y-5)^2 - 9$
so the original equation is equivalent to

$(x-4)^2+(y-5)^2 = 9 = 3^2$ which is an equation for the circle with center (4,5) and radius 3.

14. Show that the points (x, y, z) in $\mathbb{R}^3$ satisfying the equation

$$x^2+y^2+z^2-2x-4y-6z = 22$$

form a sphere of radius 6 and determine the center of the sphere.
Solution: Complete the squares in x, y and z to get $x^2-2x+1-1+y^2-4y+4-4+z^2-6z+9-9 = 22$ or
$(x-1)^2+(y-2)^2+(z-3)^2 = 22+1+4+9 = 36 = 6^2$ which is an equation of the sphere of radius 6, centered at (1, 2, 3)