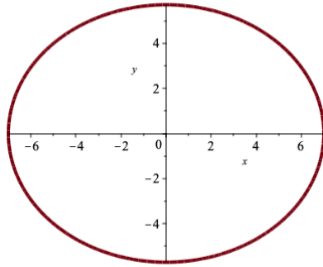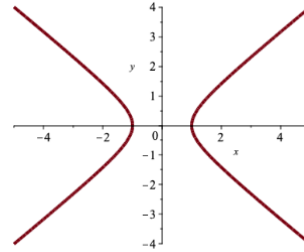1. Let $f(x,y) = 2x^2 + 3y^2$ be a real-valued function defined on the plane.
(a) Let **P** be the point (5,4) and C the level curve of f containing **P**. Identify the
nature of C: Is it a Circle? Parabola? Pair of Lines? Sketch a picture of C.
Solution: $f(5,4) = 2(5^2) + 3(4^2) = 50 + 48 = 98.\ 2x^2 + 3y^2 = 98$ is the equation of an ellipse.
Below left is a graph of the level curve.

Level Curve For Exercise 1   *Level Curve for Exercise 2*

(b) Use classic implicit differentiation or some other method to find an
equation for the line L tangent to C at **P**.
Solution: Differentiating $2x^2 + 3y^2 = 98$ with respect to x, treating y as an unspecified function of x, we
have $4x + 6y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{2x}{3y}$. Then the slope of the tangent line at (5,4) is $-\frac{2}{3} \times \frac{5}{4} = -\frac{5}{6}$. An
equation for the tangent line is $y - 4 = -\frac{5}{6}(x - 5)$.
(c) Determine the gradient vector **v** of f at **P**.
Solution: $\nabla f(x,y) = \left(f_x(x,y), f_y(x,y) \right) = (4x, 6y)$. Thus $\mathbf{v} = \nabla f(5,4) = (20,24)$.
(d) Show that this gradient vector is orthogonal to any vector lying along L.
Solution: The slope of this gradient vector is $\frac{24}{20} = \frac{6}{5}$ which is negative reciprocal of $-\frac{5}{6}$
so the vector **v** is orthogonal to the tangent line L.

2. Let $g(x,y) = 2x^2 - 3y^2$ be a real-valued function defined on the plane.
(a) Let **P** be the point (5,4) and C the level curve of g containing P. Identify the
nature of C: Is it a Circle? Parabola? Pair of Lines? Sketch a picture of C.
Solution: $g(5,4) = 2(5^2) - 3(4^2) = 50 - 48 = 2.$ Now $2x^2 - 3y^2 = 2$ is the equation of an
hyperbola. Above right is a graph of the level curve.
(b) Use classic implicit differentiation or some other method to find an
equation for the line L tangent to C at **P**.
Solution: Differentiating $2x^2 - 3y^2 = 2$ with respect to x, treating y as an unspecified function of
x, we have $4x - 6y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = \frac{2x}{3y}$. Then the slope of the tangent line at (5,4) is $\frac{2}{3} \times \frac{5}{4} = \frac{5}{6}$.
An equation for the tangent line is $y - 4 = \frac{5}{6}(x - 5)$.
(c) Determine the gradient vector **v** of g at **P**.
Solution: $\nabla f(x,y) = \left(f_x(x,y), f_y(x,y) \right) = (4x, -6y)$. Thus $\mathbf{v} = \nabla f(5,4) = (20, -24)$.
(d) Show that this gradient vector is orthogonal to any vector lying along L.
Solution: The slope of this gradient vector is $\frac{-24}{20} = \frac{-6}{5}$ which is negative reciprocal of $\frac{5}{6}$
so the vector **v** is orthogonal to the tangent line L.

3. (Williamson and Trotter) A spaceship traveling in the plane along a path such that at time $t \geq 0$, the ship is at position $g(t) = (3t^2, t^3)$. The intensity of gamma radiation at the point (x, y) in the plane is $I(x, y) = x^2 - y^2$,
    wherever $I(x, y) \geq 0$. Describe fully, using a labeled graph where appropriate, the following:
    (a) The level curve of I the ship is on at $t = 1$.
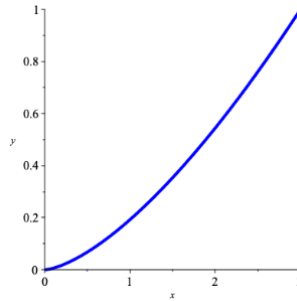   *Solution:* $g(1) = (3,1)$ is the position at $t = 1$ where $I(3,1) = 3^2 - 1^2 = 8$.
    (The level curve is the hyperbola $x^2 - y^2 = 8$
    (b) The path of the ship for $t \geq 0$.
    Solution: The rectangular coordinates of the spaceship's position at time t satisfy $x = 3t^2, y = t^3$. For $t \geq 0$, we have $t = \left(\frac{x}{3}\right)^{1/2}$ so $y = \left(\frac{x}{3}\right)^{3/2}$. The graph of this relationship is the path of the spaceship with $t = 0$ corresponding to $(x, y) = (0,0)$; we show it below:

    (c) The gradient vector of I at the ship's position when $t = 1$.
    Solution: Gradient vector of I is $\nabla f(x, y) = \left(f_x(x, y), f_y(x, y)\right) = (2x, -2y)$ so
    $$\nabla f(3,1) = (2 \times 3, -2 \times 1) = (6, -2)$$
    (d) The ship's velocity vector at $t = 1$.
    Solution: The velocity vector at time t is $g'(t) = (6t, 3t^2)$ and hence the velocity at $t=1$ is $(6, 3)$.

    (e) The time – if there is one – when the ship stops increasing its radiation risk and begins its race to safety. Does its course become more dangerous later on?
    Solution: The radiation experienced by the spaceship at time t is represented by the composite function $I\left(g(t)\right) = I(3t^2, t^3) = (3t^2)^2 - (t^3)^2 = 9t^4 - t^6 = t^4(9 - t^2)$
    which is non-negative for $0 \leq t \leq 3$. The rate at which the radiation is changing is
    $(I(g(t))' = 36t^3 - 6t^5 = 6t^3(6 - t^2)$ which is positive for $0 < t < \sqrt{6}$ and negative for $t > \sqrt{6}$. The maximum radiation occurs at $t = \sqrt{6}$ where level is $\left(\sqrt{6}\right)^4 \left(9 - \left(\sqrt{6}\right)^2\right) =$
    $(36)(9 - 6) = 108$. The spaceship begins its race to safety at $t = \sqrt{6}$ from which time it experiences less and less radiation until it drops to zero at $t = 3$. The graph of the radiation as function of time is below: