MATH 223

Some Notes on Assignment 32

Exercises 27, 29a and 31 of Chapter 8.

27: Let $\sigma(s,t) = \begin{pmatrix} a\cos s \sin t \\ a \sin s \sin t \\ a\cos t \end{pmatrix}, 0 \le s \le 2\pi, 0 \le t \le 2\pi$. Then σ parameterizes a surface S in $\mathbb{R}^3$. Describe S and find its surface area.

Solution: With $x = a\cos s \sin t, y = a \sin s \sin t$
$a\cos t, z = a\cos t$, we have

$$\begin{aligned} x^2+y^2+z^2 &= a^2\cos^2 s\sin^2 t + a^2 \sin^2 s \sin^2 t + a^2\cos^2 \\ &= a^2(\cos^2 s + \sin^2 s)\sin^2 t + a^2\cos^2 t = a^2 \sin^2 t + a^2 \cos^2 t \\ &= a^2(\sin^2 t + \cos^2 t) = a^2 \end{aligned}$$

so S is the sphere of radius a centered at the origin. As t progresses from 0 to 0 to π, we move from one pole $(z = a)$ to the equation to the other pole $(z = -a)$. The entire sphere is generated for t between 0 and π.
To find the surface area, we use

$$Area(S) = \int_D |\sigma_s(s,t) \times \sigma_t(s,t)|\, ds\, dt$$

To conserve space, we'll write $\sigma(s,t)$ horizontally as $\sigma(s,t) = (a\cos s \sin t,\ a \sin s \sin t a \cos t,\ a\cos t)$ so $\sigma_s(s,t) = (-a\sin s \sin t,\ a\cos s \sin t,\ 0)$ and $\sigma_t(s,t) = (a\cos s\cos t,\ a \sin s \cos t,\ -a \sin t)$.
Then $\sigma_s(s,t)\times\sigma_t(s,t) = \big(-a^2\cos s \sin^2 t, -a^2 \sin s \sin^2 t, -a^2 \sin t \cos t\big)$ and $|\sigma_s(s,t)\times\sigma_t(s,t)| = a^2 \sin t$.

Thus Area$(S) = \int_{s=0}^{s=2\pi}\int_{t=0}^{t=\pi} a^2 \sin t\, dt\, ds = \int_{s=0}^{s=2\pi} 2a^2\, ds = 4a^2\pi$.

29a: Suppose $f : R^3 \to R^1$ is a continuous real-valued function and S is a smooth surface in $\mathbf{R}^3$ parameterized by $\sigma(s,t) : A \to \mathbf{R}^3$ for some set A in the plane. Then the **surface integral of the scalar function** f **over** S is given by

$$\iint_S f\, d\sigma = \iint_A f(\sigma(s,t)|\sigma_s(s,t)\times\sigma_t(s,t)|\, ds\, dt$$

Find $\iint_S f\, d\sigma$ if $f(x,y,z) = \frac{x}{\sqrt{4y+5}} + z$ and $\sigma(s,t) = (s, t^2-1, t), 0 \le s \le 1, 0 \le t \le 1$.

Solution: $f(\sigma(s,t) = f(s, t^2-1, t) = \frac{s}{\sqrt{4(t^2-1)+5}} + t = \frac{s}{\sqrt{4t^2=1}} + t$.

We also have

$$|\sigma_s(s,t)\times\sigma_t(s,t)| = |(1,0,0)\times(0,2t,1)| = |0,-1,2t| = \sqrt{4t^2+1}$$

$$\text{Thus } \sigma(s,t)|\sigma_s(s,t)\times\sigma_t(s,t)| = s + t\sqrt{4t^2+1}$$

$$\text{and } \iint_S f\, d\sigma = \int_{t=0}^{t=1}\int_{s=0}^{s=1} s + t\sqrt{4t^2+1}\, ds\, dt = \int_{t=0}^{t=1} \frac{1}{2} + t\sqrt{4t^2+1}\, dt = \frac{5}{12}(1+\sqrt{5})$$

31: Determine $\int_S \mathbf{F}\cdot d\mathbf{S}$ if $\mathbf{F}(x,y,z) = (x, 2y, 3z)$ and $\sigma(s,t) = (st, s+t, 2s-t), 0 \le s \le 1, 0 \le t \le 1$.

Solution: Use $\int_S \mathbf{F}\cdot d\mathbf{S} = \int_D \mathbf{F}(\sigma(s,t)\cdot(\sigma_s(s,t)\times\sigma_t(s,t)\, ds\, dt$.

Here $\mathbf{F}(\sigma(s,t) = \mathbf{F}(st, s+t, 2s-t) = (st, 2s+2t, 6s-3t)$ while $\sigma_s(s,t) = (t,1,2)$ and $\sigma_t(s,t) = (s,1,-1)$ so $\sigma_s(s,t)\times\sigma_t(s,t) = (-3, 2s+t, -s+t)$ and hence $\mathbf{F}(\sigma(s,t)\cdot(\sigma_s(s,t)\times\sigma_t(s,t) = 12st - 2s^2 - t^2$.

Thus $\int_S \mathbf{F}\cdot d\mathbf{S} = \int_{t=0}^{t=1}\int_{s=0}^{s=1} 12st - 2s^2 - t^2\, ds\, dt = \int_{t=0}^{t=1} -\frac{2}{3} + 6t - t^2\, dt = 2$.