MATH 223

Some Notes on Assignment 28

Exercises 17ace, 18ad, 19a, 20, and 21 of Chapter 7.

17ace: Set up, but do not evaluate, an integral which represents the arc length of each of the following curves:
(a) $f(x) = x^2, 0 \le x \le 1$ (c) $f(x) = e^{2x}, 0 \le x \le 2$ (e) $\mathbf{f}(t) = (\sin t, \cos t), 0 \le t \le \pi/2$

Solutions We use

$$\mathcal{L}(\gamma) = \int_a^b |\mathbf{g}'(t)|\, dt.$$

(a) Let $g(t) = (t, t^2), 0 \le t \le 1$ so $g'(t) = (1, 2t)$. Then $|g'(t)| = \sqrt{1+4t^2}$. Here $\mathcal{L}(\gamma) = \int_a^b |\mathbf{g}'(t)|\, dt = \int_0^1 \sqrt{1+4t^2}\, dt$.
(c) Let $g(t) = (t, e^{2t}), 0 \le t \le 2$ so $g'(t) = (1, 2e^{2t})$. Then $|g'(t)| = \sqrt{1+4e^{4t}}$. Here $\mathcal{L}(\gamma) = \int_0^2 \sqrt{1+4e^{4t}}\, dt$
(e) Let $g(t) = \mathbf{f}(t)$ so $g'(t) = (\cos t, -\sin t)$ which has length 1 so Here $\mathcal{L}(\gamma) = \int_0^{\pi/2} 1\, dt = \pi/2$. [Note: we could simply observe that the curve is one quarter of the unit circle and we know the circumference of a circle.]

18ad: Determine the length of each of these curves using

$$\mathcal{L}(\gamma) = \int_a^b |\mathbf{g}'(t)|\, dt.$$

(a) $f(x) = x^{2/3}, 1 \le x \le 3$
Solution: Let $g(t) = (t, t^{2/3})$ so $g'(t) = \left(1, \frac{2}{3}t^{-1/3}\right)$. Then $|g'(t)| = \sqrt{1+\frac{4}{9}t^{-2/3}} = \sqrt{1+\frac{4}{9t^{2/3}}}$

$$= \frac{\sqrt{4+9t^{2/3}}}{9t^{2/3}} = \frac{\sqrt{4+9t^{2/3}}}{3t^{1/3}} \text{ so } \mathcal{L}(\gamma) = \int_1^3 \frac{\sqrt{4+9t^{2/3}}}{3t^{1/3}}\, dt$$

Make the change of variable $u = 4+9t^{2/3}$. Then $\frac{1}{6}du = \frac{1}{t^{1/3}}dt$ so that

$$\int \frac{\sqrt{4+9t^{2/3}}}{3t^{1/3}}\, dt = \int \frac{\sqrt{u}}{18}\, du = \frac{u^{3/2}}{27} = \frac{1}{27}\left(4+9t^{2/3}\right)^{3/2}$$

and the value of the definite integral is $\frac{1}{27}\left((4+9\,3^{2/3})^{3/2} - 13^{3/2}\right)$

(d) $\mathbf{f}(t) = (t+\sin t, \cos t), 0 \le t \le \pi/2$
Solution: Use $g(t) = (t+\sin t, \cos t)$ which has $g'(t) = (1+\cos t, -\sin t)$ so that $|g'(t)| = \sqrt{1+2\cos t+\cos^2 t+\sin^2 t} = \sqrt{2+2\cos t} = \sqrt{2}\sqrt{1+\cos t} = \sqrt{2}\sqrt{1+\cos(2\frac{t}{2})} = \sqrt{2}\sqrt{2\cos^2\frac{t}{2}} = \sqrt{2}\sqrt{2}\sqrt{\cos^2\frac{t}{2}} = 2\cos\frac{t}{2}$. Then

$$\mathcal{L}(\gamma) = \int_0^{\frac{\pi}{2}} 2\cos\frac{t}{2}\, dt = 4 sin\frac{\pi}{4} - 4\sin 0 = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}.$$

19a Winding Number I; Let γ be a closed curve in the plane with parametrization $\mathbf{g}(t), a \le t \le b$ so that $g(a) = g(b)$. Suppose γ does not pass through the origin; that is, there is no t such that $\mathbf{g}(t) = (0,0)$. The **winding number of γ about the origin** $w(\gamma)$ is

$$w(\gamma) = \frac{1}{2\pi}\int_\gamma \mathbf{F} \text{ where } \mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

For each of the following, sketch the curve γ, compute the winding number, and verify that $w(\gamma)$ describes the number of times the curve circles around the origin in a counterclockwise direction:

(a) γ has parametrization $\mathbf{g}(t) = (3\cos t, 3\sin t), 0 \le t \le 2\pi$

*Solution:*The curve γ is the circle of radius 3, centered at the origin, traced out once in a counterclockwise direction. Since $x = 3\cos t, y = 3\sin t$, we have $x^2 + y^2 = 3^2\cos^2 t + 3^2 \sin t = 9$ which yields $\mathbf{F}(x,y) = \frac{-3\sin t}{9}, \frac{3\cos t}{9} = (-\frac{1}{3}\sin t, \frac{1}{3}\cos t$ and $g'(t) = (-3\sin t, 3\cos t)$. Thus

$$w(\gamma) = \frac{1}{2\pi}\int_0^{2\pi}(-\frac{1}{3}\sin t, \frac{1}{3}\cos t)\cdot(-3\sin t, 3\cos t)dt = \frac{1}{2\pi}\int_0^{2\pi}\sin^2 t + \cos^2 t dt = \frac{1}{2\pi}\int_0^{2\pi} 1 dt = \frac{1}{2\pi}2\pi = 1$$

20: If the equation $y = f(x), a \le x \le b$ for a continuously differentiable function f defines a curve C in the plane, show that the length of the curve C is

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

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Solution: Let $g(t) = (t, f(t))$. so $g'(t) = (1, f'(t))$ and $|g'(t)| = \sqrt{1 + [f'(t)]^2}$ and thus the length of the curve is $\int_a^b \sqrt{1 + [f'(t)]^2}\, dt$.

21: The equation $r = f(\theta), a \le \theta \le$ describes a curve in polar coordinates. If f is a continuously differentiable function, show that curve's length is

$$\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

Hint: $(x,y) = (r\cos\theta, r\sin\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta), a \le \theta \le b$ parametrizes the curve.
Solution: Let $g(\theta) = f(\theta)\cos\theta, f(\theta)\sin\theta)$ so $g'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta)$

$$\begin{aligned}
|g'(\theta)| &= \sqrt{[f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2} \\
&= \sqrt{[f'(\theta)]^2\cos^2\theta - 2f(\theta)d'(\theta)\sin\theta\cos\theta + [f(\theta)]^2 + [f'(\theta)]^2\sin^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta + [f(\theta)]^2\cos^2\theta} \\
&= \sqrt{[f'(\theta)]^2(\cos^2\theta + \sin^2\theta) + [f(\theta)]^2(\cos^2\theta + \sin^2\theta)} \\
&= \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}.
\end{aligned}$$

Thus

$$\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$