MATH 223

Some Notes on Assignment 27

Exercises 9ac, 10df, 13 and 16 of Chapter 7.

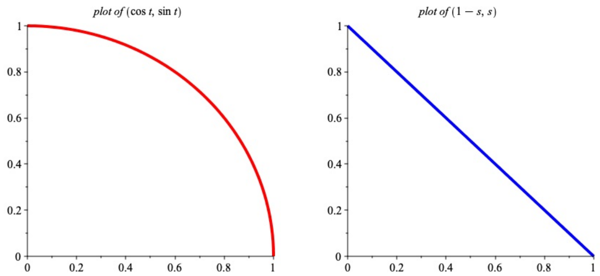
9ac: Compute the indicated line integrals $\int_\gamma \mathbf{F}$ for the given vector fields and parametrizations $\mathbf{g}$ for γ

(a) $\mathbf{F} = (x, 2y, 3z), \mathbf{g}(t) = (t, t, t), 0 \le t \le 2$
Solution: $\int_\gamma \mathbf{F} = \int_0^2 (t, 2t, 3t) \cdot (1, 1, 1)\, dt = \int_0^2 6t\, dt = \left[3t^2\right]_0^2 = 12.$

(b)$\mathbf{F} = \left(\frac{1}{x^2+y^2}, \frac{1}{x^2+y^2}\right), \mathbf{g}(t) = (\cos t, \sin t), 0 \le t \le \pi$
Solution: $\int_\gamma \mathbf{F} = \int_0^\pi \left(\frac{1}{\cos^2 t+\sin^2 t}, \frac{1}{\cos^2 t+\sin^2 t}\right) \cdot (-\sin t, \cos t)\, dt = \int_0^\pi (1, 1) \cdot (-\sin t, \cos t)\, dt = \int_0^\pi -\sin t + \cos t\, dt = [\cos t + \sin t]_0^\pi = (-1+0) - (1+0) = -2.$

10df: Sketch the curves γ_1 and γ_2 given respectively by the parametrizations $\mathbf{g_1}(t) = (\cos t, \sin t), 0 \le t \le \pi/2$ and $\mathbf{g_2}(s) = (1-s, s), 0 \le s \le 1$.

Solution:



Then determine the line integral of the vector field $\mathbf{F}$ along both γ_1 and γ_2 for each of the following vector fields:

(d) $\mathbf{F}(x, y) = (x, y)$
Solution: $\int_{\gamma_1} \mathbf{F} = \int_0^{\pi/2} (\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt = \int_0^{\pi/2} 0\, dt = 0$
$\int_{\gamma_2} \mathbf{F} = \int_0^1 (1-s, s) \cdot (-1, 1)\, ds = \int_0^1 -1+s+s\, ds = \int_0^1 2s - 1\, ds = \left[s^2 - s\right]_0^1 = (1-1) - (0-0) = 0.$

(f) $\mathbf{F}(x, y) = (xy^2, x^2y)$
Solution: $\int_{\gamma_1} \mathbf{F} = \int_0^{\pi/2} (\cos t \sin^2 t, \sin t \cos^2 t) \cdot (-\sin t, \cos t) dt = \int_0^{\pi/2} -sin^3 t \cos t + \cos^3 t \sin t dt = \left[-\frac{\sin^4 t}{4} + -\frac{\cos^4}{4}\right]_0^{\pi/2} = [-1-0] - [-0-1] = -1+1 = 0$
$\int_{\gamma_2} \mathbf{F} = \int_0^1 \left((1-s)s^2, (1-s)^2 s\right) \cdot (-1, 1)\, ds = \int_0^1 2s^3 - 3s^2 + s\, ds = \left[\frac{s^4}{2} - s^3 + \frac{s^2}{2}\right]_0^1 = 0.$

11: Calculate the line integral of the gradient of the Cobb-Douglas function $f(K, L) = 2K^{2/3}L^{1/3}$ of Example 2 along

(a) the straight line between (1,1) and (27, 64)
Solution: By the Fundamental Theorem, the line integral equals $f(27, 64) - f(1, 1) = 2(9)(4) - 2(1)(1) = 70$

(b) the level curve where $f(K, L) = 1000$. *Solution:* 0

13: If $\mathbf{F}$ is the vector field $F(S, I, R) = (-\beta SI, \beta SI - rI, rI)$, described in the SIR Epidemic Model (Example 4), find the line integral of $\mathbf{F}$ along the curve γ with parametrization $\mathbf{g}(t) = (t^2, t, t^2), 100 \le t \le 200$.
Solution: $F(t^2, t, t^2) = (-\beta t^2 t, \beta t^2 t - rt, rt) = (-\beta t^3, \beta t^3 - rt, rt)$ and $g'(t) = (2t, 1, 2t)$. Thus $F(g(t)) \cdot g'(t)) = -2\beta t^4 + \beta t^3 - rt + 2rt^2$ the line integral is $\int_{100}^{200} -2\beta t^4 + \beta t^3 - rt + 2rt^2 dt = \left[-\frac{2}{5}\beta t^5 + \frac{1}{4}\beta t^4 + \frac{2}{[3}2rt^3 - \frac{1}{2}rt^2\right]_{100}^{200} = -123625000000\beta + \frac{13955000}{3}r.$

16: Let $\mathbf{F}$ be the vector field defined by $\mathbf{F}(x, y) = (-y^2, 2x^2)$.

(a) Sketch the vector field.
Solution: See the Assignment 27 sheet.

(b) Find two different curves connecting the points (0,0) and (2,4) so that the line integrals of $\mathbf{F}$ along the curves are different.

Solution: Let g_1 be the straight line segment $g_1(t) = (t, 2t), 0 \leq t \leq 2$ with $g_1'(t) = (1, 2)$. Then $F(g_1(t)) \cdot g_1'(t) = (-4t^2, 2t^2) \cdot (1, 2) = -4t^2 + 4t^2 = 0$ so $\int_{g_1} \mathbf{F} = 0$

Let $g_2(t) = (t, t^2), 0 \leq t \leq 2$ with $g_2'(t) = (1, 2t)$. Then $F(g_2(t)) \cdot g_1'(t) = (-t^4, 2t^2) \cdot (1, 2t) = -t^4 + 4t^3$ so $\int_{g_2} \mathbf{F} = \int_0^2 -t^4 + 4t^3 \, dt = \frac{48}{5}$

(c) Find a closed curve γ in the plane where the starting point is the same as the ending point but $\int_\gamma \mathbf{F}$ is not zero.

Solution: Let γ be the path that starts at (0,0), runs along g_1 to (2,4) and then runs back along g_2 to (0,0).

(d) Explain why parts (b) and (c) each allow you to conclude that $\mathbf{F}$ is not a gradient field.

Solution: For a gradient field, the line integrals between the same pair of points should be equal and the integral around a closed curve should be 0 if the vector field is continuous everywhere.