MATH 223

Some Notes on Assignment 22

Exercises 8, 9, 10, 11 and 12 in Chapter 6

8: Determine the integral of $f(x, y) = x^2 + y^2$ over the region bounded by the x-axis and the top half of the unit circle centered at the origin.

Solution: Figure 1 shows the region. Carving thee region into vertical lines, we see that or each x between -1 and 1, a vertical segment runs from the horizontal axis up to the semicircle; that is, $y = 0$ to $y = \sqrt{1-x^2}$. Thus the value of the integral is

$$\int_{x=-1}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} x^2 + y^2\, dy\, dx = \int_{x=-1}^{x=1}\left[x^2y + \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{1-x^2}} dx = \int_{x=-1}^{x=1} x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3}\, dx$$

The last integral can be solved in a variety of ways including integration by parts, the substitution $x = \sin\theta$, and the recognition that $\int_{-1}^{1}\sqrt{1-x^2}\, dx$ is the area $\pi/2$ of a semicircle of radius 1. The indefinite integral is $\frac{x\sqrt{1-x^2}}{4} - \frac{x(1-x^2)^{3/2}}{4} + \frac{\arcsin x}{8}$. The first two terms yield 0 when compute the definite inetgral so the value of the original iterated integral is $\frac{1}{8}(\arcsin 1 - \arcsin -1) = \frac{1}{8}(\pi/2 - (-pi/2)) = \pi/4$.

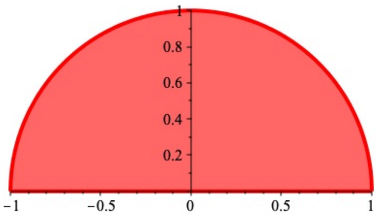


Figure 1: Region of Exercise 8

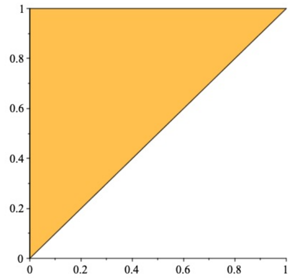


Figure 2: Region of Exercise 9

9: Find the value of the integral of $f(x, y) = x^2 + y^2$ over the region enclosed by the triangle with vertices (0,0), (0,1), and (1,1).

Solution: Figure 2 shows the region. Each horizontal slice runs from $x = 0$ to $x = y$ and we have a horizontal slice for each y from 0 to 1. Thus we can evaluate the integral as

$\int_{y=0}^{y=1}\int_{x=0}^{x=y} x^2 + y^2\, dx\, dy = \int_{y=0}^{y=1}\left[\frac{x^3}{3} + xy^2\right]_{x=0}^{x=y} dy = \int_{y=0}^{y=1}\frac{y^3}{3} + y^3\, dy = \int_{y=0}^{y=1}\frac{4}{3}y^3\, dy = \left[\frac{y^4}{3}\right]_0^1 = \frac{1}{3}$

10: Evaluate the integral of $2x+3y+4z$ over the region enclosed by the tetrahedron with vertices (0,0,0), (0,0,3), (0,2,0), and (1,0,0).

Solution: Figure 3 shows the tetrahedron. Three of its four sides are the coordinate planes and the fourth is the plane with equation $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$.. You can set up the order of integration in 6 possible ways. We'll do it as $\iiint 2x + 3y + 4z\, dz\, dy\, dx$. Figures 4, 5 and 6 display the xy, xz and yz slices respectively.

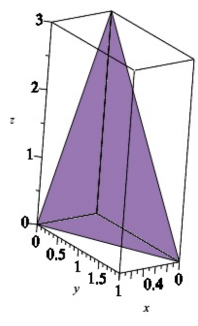
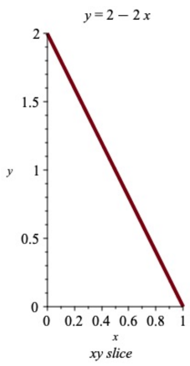
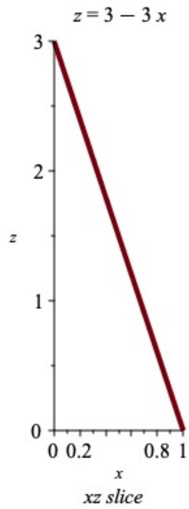


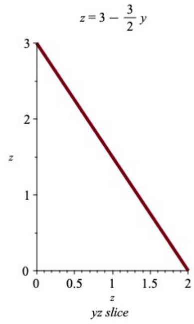
Figure 3



xy- slice; $z = 0$
$y = 2 - 2x$
Figure 4



xz-slice: $y = 0$
$z = 3 - 3x$
Figure 5



yz-slice:x $= 0$
$z = 3 - \frac{3}{2}y$
Figure 6

Solving the plane equation for z, we have $z = 3 - 3x - \frac{3}{2}y$ and from Figure 4, we have $y = 2 - 2x$ so the the triple integral is

$$\int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} \int_{z=0}^{z=3-3x-\frac{3}{2}y} (2x + 3y + 4z)\, dz\, dy\, dx$$

which equals

$$\int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} 12x^2 + 6xy - 30x - 9y + 18\, dy\, dx = \int_{x=0}^{x=1} -12x^3 + 42x^2 - 48x + 18\, dx = 5$$

11: Determine the volume bounded by the coordinate axes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution: Proceed as in Exercise 10. The vertices of the region are $(a, 0, 0), (0, b, 0), (0, 0, c)$

Volume is

$$\int_{x=0}^{x=a} \int_{y=0}^{y=b-\frac{b}{a}x} \int_{z=0}^{z=c-\frac{c}{a}x-\frac{c}{b}y} 1\; z\, dy\, dx = \frac{abc}{6}$$

12: Find the volume of the solid bounded by the surfaces $y^2 + z^2 = 4ax, x = 3a$, and $y^2 = ax$.

The equations $x = 3a$ and $y^2 = ax$ define a figure in the plane bounded by a parabola and a straight line segment. See Figure 7 in red below. The line segment and the parabola intersect at the points $(3a, \pm\sqrt{3}a)$. A double integral over this region would be written as

$$\int_{x=0}^{x=3a} \int_{y=-\sqrt{ax}}^{\sqrt{ax}} f(x, y)\, dy\, dx$$

if we imagine the region carved into vertical slices.

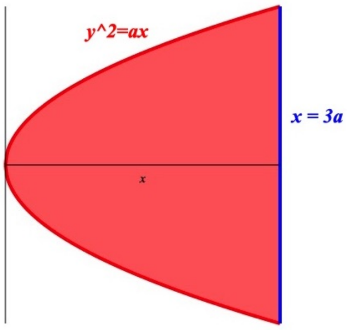


Figure 7

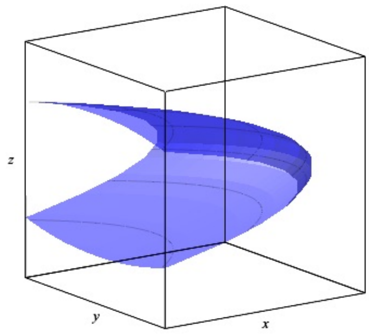


Figure 8

Figure 8 displays a graph of the surface defined by $y^2 + z^2 = 4ax$.

We can solve the remaining equation $y^2 + z^2 = 4ax$ for z in terms of x and y:

$$z = \pm\sqrt{4ax - y^2}$$

. To find the volume of the solid, we set up the triple integral

$$V = \int_{x=0}^{x=3a} \int_{y=-\sqrt{ax}}^{\sqrt{ax}} \int_{z=-\sqrt{4ax-y^2}}^{z=\sqrt{4ax-y^2}} 1\, dz\, dy\, dx$$

Carrying out the integral with respect to z, we have

$$V = \int_{x=0}^{x=3a} \int_{y=-\sqrt{ax}}^{\sqrt{ax}} 2\sqrt{4ax - y^2}\, dy\, dx$$

One way to do the integration with respect to y is to let $A = 4ax$ and use the trig substitution $\sin\theta = y/\sqrt{A}$ which converts

$$\int \sqrt{A - y^2}\, dy \text{ to } \int A\cos^2\theta\, d\theta = \frac{A}{2}\left[\theta + \sin\theta\cos\theta\right] = \frac{1}{2}\left[A \arcsin\frac{y}{\sqrt{A}} + y\sqrt{A - y^2}\right]$$

Substituting $4ax$ for A, the last expression becomes

$$\frac{1}{2}\left[4ax \arcsin\frac{y}{2\sqrt{ax}} + y\sqrt{4ax - y^2}\right]$$

Now we evaluate this expression at $y = \sqrt{ax}$ and $y = -\sqrt{ax}$ and compute the difference. At $y = \sqrt{ax}$, we obtain

$$\begin{aligned}
\frac{1}{2}\left[4axx \arcsin\frac{\sqrt{ax}}{2\sqrt{ax}} + \sqrt{ax}\sqrt{4ax - ax}\right] &= \frac{1}{2}\left[4ax \arcsin\frac{1}{2} + \sqrt{ax}\sqrt{3ax}\right] \\
&= \frac{1}{2}\left[4ax\frac{\pi}{6} + \sqrt{ax}\sqrt{ax}\sqrt{3}\right] \\
&= \frac{1}{2}ax\left[\frac{2\pi}{3} + \sqrt{3}\right]
\end{aligned}$$

The value at $y = -\sqrt{ax}$ is the negative of this value. Hence the volume becomes

$$V = \int_{x=0}^{x=3a} ax\left[\frac{2\pi}{3} + \sqrt{3}\right] dx = \frac{9}{2}a^3\left[\frac{2\pi}{3} + \sqrt{3}\right] = a^3\left[3\pi + \frac{9}{2}\sqrt{3}\right]$$