MATH 223

Some Notes on Assignment 19

Exercises 18ad, 19ac, 25 and 26 in Chapter 5

18: Find, if they exist, the highest and lowest points on each of the following surfaces

- (a) $z = x^2 - xy + 4x + y^2 + 6y + 25 + 1/3$
- (d) $z = x^2 + xy + 3x + 2y$

Solution: If we write each equation as a function of x and y such that $f(x, y) = z$, then each function is real-valued and continuous on all of $\mathbb{R}^2$. If there is a highest or lowest point of the function, then the gradient will be zero when evaluated at that point.

a) Because $f(x, y)$ is continuous on all of $\mathbb{R}^2$, if it does achieve a maximum or minimum somewhere, then the gradient will be the zero vector at that point. The gradient of f is $\nabla f = (2x - y + 4, 2y - x + 6)$. Wherever the gradient is the zero vector we have

$$2x - y + 4 = 0 \Rightarrow 2x + 4 = y$$

$$2y - x + 6 = 0 \Rightarrow 2(2x + 4) - x + 6 = 0 \Rightarrow x = \frac{-14}{3}.$$

Substituting this value for x into either of the partial derivatives we find that f has exactly one critical point at $\left(\frac{-14}{3}, \frac{-16}{3}\right)$. Now does f have an extreme value at this point? Is is a minimum? A maximum? A saddle point? We will need to use the second derivative test in order to determine its nature. The derivative of the gradient of f is a two by two Hessian matrix in which each row is the partial derivatives of the components of the gradient.

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

To determine whether or not this is a positive or negative definite matrix we inspect the values of $\mathbf{x} \cdot H\mathbf{x}$ for an arbitrary vector $\mathbf{x} = (x_1, x_2)$.

$$\mathbf{x} \cdot H\mathbf{x} = \mathbf{x} \cdot (2x_1 - x_2, 2x_2 - x_1)$$

$$\mathbf{x} \cdot H\mathbf{x} = 2(x_1^2 - x_1x_2 + x_2^2)$$

If x_1 and x_2 have different signs, then the right hand side of this equation is positive. If x and y have the same sign and are equal then we have $x_1^2 - x_1x_2 + x_2^2 = x_1^2$. In the case that x_1 and x_2 have the same sign and (without loss of generalization) $x_1 < x_2$ we have

$$x_1^2 - x_1x_1 + x_1^2 < x^2 - x_1x_2 + x_2^2$$

$$0 < x_1^2 < x_1^2 - x_1x_2 + x_2^2.$$

$\mathbf{x} \cdot H\mathbf{x}$ is then positive for all non zero vectors $\mathbf{x}$, and H is positive definite. Theorem 5.6.4 promises that if a function f has continuous third order partial derivatives and the Hessian matrix of f at a critical point is positive-definite, then the critical point is a relative minimum. The third order partial derivatives of f are all 0, and the Hessian matrix at $\left(\frac{-14}{3}, \frac{-16}{3}\right)$ is positive definite; thus, this point is a relative minimum. Because f is continuous and there are no other critical points, f achieves its minimum value at this point. The minimum value of f is $f\left(\frac{-14}{3}, \frac{-16}{3}\right) = 0$.

d) If we write z as a function of x and y we have $f(x, y) = x^2 + xy + 3x + 2y$. The gradient of f is $\nabla f = (2x + y + 3, x + 2)$. Letting each partial derivative of f equal 0 and solving for x and y we find that $(-2, 1)$ is a critical point of f. Furthermore, $f(-2, 1) = -2$. To determine whether or not this point is an extreme, we need to investigate the Hessian matrix of f evaluated at the critical point. The Hessian of f is

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

at all points in the domain of f. This matrix has eigenvalues, λ, wherever the matrix

$$\begin{pmatrix} 2 - \lambda & 1 \\ 1 & 0 - \lambda \end{pmatrix}$$

has a determinant of 0. Applying the Quadratic formula to the determinant of the matrix we find $\lambda = 1 \pm \sqrt{2}$. Thus, the Hessian matrix has one positive eigenvalue and one negative eigenvalue; it is neither positive definite nor negative definite. The critical point $(-2, 1)$ is a saddle point and not an extreme of f. As there are no other critical points of f and the function is continuous for all (x, y), there must be no highest and lowest values of $f(x, y) = z$.

19: Graph level curves for each of the following real-valued function of two variables and then determine and classify each of the critical points: (a). $f(x, y) = x^3 - y^3$ and (c) $f(x, y) = \frac{1}{e^{x^2+Y^2}}$

Solution: (a) A critical point of f is any point (x, y) such that $\nabla f(x, y) = (0, 0)$. The gradient of f is $\nabla f(x, y) = (3x^2, -3y^2)$. The only point at which the gradient is the zero vector is the origin; thus, the origin is the only critical point. To determine whether f achieves a local maximum, minimum or neither at this point we must inspect the Hessian matrix of f. Taking the partial derivatives with respect to x and y of each function in the gradient we have

$$H = \begin{pmatrix} 6x & 0 \\ 0 & -6y \end{pmatrix}.$$

Evaluated at the origin the Hessian of f is

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For any arbitrary 2 by 1 vector $\mathbf{x}$, $x \cdot H\mathbf{x} = \mathbf{0}$, so H is neither positive nor negative definite. The function f then has a saddle point at the origin.

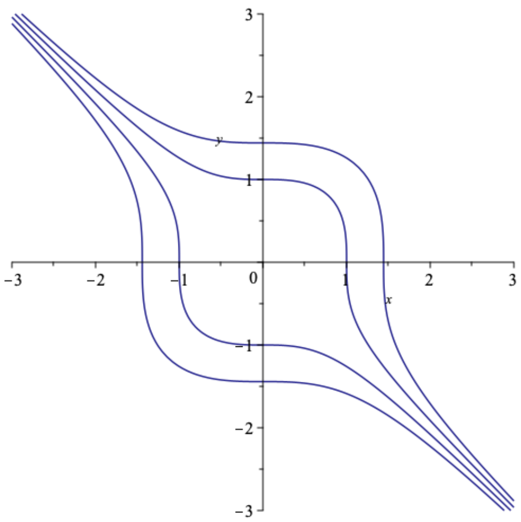

Figure 1: Level curves for the function $f(x, y) = x^3 - y^3$.

(c) If $f(x, y) = \frac{1}{e^{x^2+y^2}}$ then the gradient of f is $\nabla f = \left(\frac{-2x}{e^{e+y^2}}, \frac{-2y}{e^{x^2+y^2}}\right)$. We can see that the partial derivative f_x will only be 0 when $x = 0$ and the partial derivative f_y will only be 0 when $y = 0$; thus, the only critical point of f is the origin. To find the Hessian matrix of f we must find all 4 second order partial derivatives of f. Applying the quotient rule to f_x we find

$$f_{xx} = \frac{e^{x^2+y^2}(-2) - (-2x)e^{x^2+y^2}(2x)}{e^{2(x^2+y^2)}}$$

$$f_{xx} = \frac{4x^2 - 2}{e^{x^2+y^2}}$$

Differentiating f_x with respect to y we have

$$f_{xy} = \frac{4xy}{e^{2(x^2+y^2}}$$

Notice that if we substitute x for y in the partial derivative f_y we get $f_y = f_x$. We can then substitute y for x in the two second order partial derivatives above to find f_{yy} and f_{yx}.

$$f_{yy} = \frac{4y^2 - 2}{e^{x^2+y^2}}$$

$$f_{yx} = \frac{4xy}{e^{2(x^2+y^2}}$$

The Hessian matrix of f is

$$H = \begin{pmatrix} \frac{4x^2-2}{e^{x^2+y^2}} & \frac{4xy}{e^{2(x^2+y^2}} \\ \frac{4y^2-2}{e^{x^2+y^2}} & \frac{4xy}{e^{2(x^2+y^2}} \end{pmatrix}.$$

Evaluating the Hessian at the origin we find

$$H = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The eigenvalues of this matrix are all values, λ, such that $det(H - \lambda I) = 0$.

$$det(H - \lambda I) = 0 = (-2 - \lambda)^2 \Rightarrow \lambda = -2$$

The only eigenvalue of H is -2. Hessian matrices with only negative eigenvalues are negative definite, so H is negative definite and the origin is a local maximum of f.

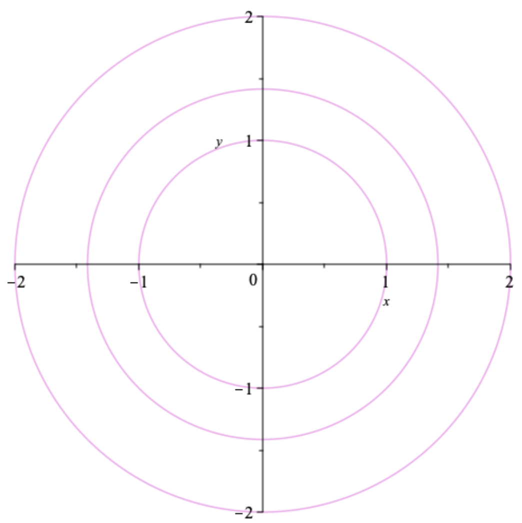

Figure 2: Level curves for the function $f(x, y) = \frac{1}{e^{x^2+y^2}}$.

25: Abigail, Anne and Sasha have an older brother Eli who is an applied mathematician. He realizes that his siblings' problems are specific instances of a more general problem

$$\text{Maximize } f(x, y) = kx^\alpha y^\beta \text{ subject to } ax + by = c$$

where a, b, c, k, α, and β are given positive constants with $\alpha + \beta = 1$. Eli uses Lagrange's method to solve the problem. What does his solution look like?
Solution: Eli can solve the general case of the maximization problem using the same technique as was applied in the specific case for each of his sisters. The function he is looking to maximize is $f(x, y) = kx^\alpha y^\beta$ with constraint $g(x, y) = ax + by - c = 0$. Applying the method of Lagrange Multipliers we have $F(x, y, \lambda) = kx^\alpha y^\beta - \lambda(ax + by - c)$. The gradient of this function is

$$\nabla F = (\alpha k y^\beta x^{-\beta} - \lambda a, \beta k x^\alpha y^{-\alpha} y^{-\alpha} - \lambda b, c - ax - by).$$

F has a critical point when $\nabla F = (0, 0, 0)$ and

$$1. \quad \alpha k y^\beta x^{-\beta} - \lambda a = 0,$$

$$2. \quad \beta k x^{\alpha} y^{-\alpha} y^{-\alpha} - \lambda b = 0,$$

$$3. \quad c - ax - by = 0.$$

Multiply equation 1 by b and equation 2 by a we have

$$b\alpha k y^{\beta} x^{-\beta} = a\beta k x^{\alpha} y^{-\alpha} y^{-\alpha}.$$

Making use of the fact that $\alpha + \beta = 1$ we simplify this equation to find

$$b\alpha y = a\beta x$$

$$y = \frac{a\beta}{b\alpha} x.$$

Substituting this value into equation 3 we can solve for x in terms of the given constants.

$$c - ax - b\frac{a\beta}{b\alpha} x = 0$$

$$ax + \frac{a\beta x}{\alpha} = c$$

$$x\left(\frac{a\alpha}{\alpha} + \frac{a\beta}{\alpha}\right) = c$$

$$x\left(\frac{a(\alpha+\beta)}{\alpha}\right) = c$$

$$x = \frac{c\alpha}{a} \Rightarrow y = \frac{c\beta}{b}.$$

The functions $F(x, y, \lambda)$ has a critical point when $x = \frac{c\alpha}{a}$ and $y = \frac{c\beta}{b}$; thus f is maximized at this point.

26: Use the method of Lagrange multipliers to investigate solving the problem

$$\text{Maximize } f(x, y, z) = kx^{\alpha} y^{\beta} z^{\gamma} \text{ subject to } ax + by + cz = d$$

where $a, b, c, d, k, \alpha, \beta$, and γ are given positive constants with $\alpha + \beta + \gamma = 1$. *Solution:* We can apply Lagrange Multipliers to a function of three variables using the same method as in the two variable case. First, we must recognize that the function being maximized if $f(x, y, z) = kx^{\alpha} y^{\beta} z^{\gamma}$ and the constraint on x, y, and z is $ax + by + cz = d$. Just as in the two variable case we must form a new function $F(x, y, z, \lambda)$ which has a critical point at the maximum of f.

$$F(x, y, z, \lambda) = kx^{\alpha} y^{\beta} z^{\gamma} - \lambda(ax + by + cz - d)$$

$$\nabla F = \left(\alpha k x^{\alpha-1} y^{\beta} z^{\gamma} - \lambda a, \beta k x^{\alpha} y^{\beta-1} z^{\gamma} - \lambda b, \gamma k x^{\alpha} y^{\beta} z^{\gamma-1} - \lambda c, d - ax - by - cz\right)$$

Wherever F has a critical point we have

$$1. \quad \alpha k x^{\alpha-1} y^{\beta} z^{\gamma} = \lambda a$$

$$2. \quad \beta k x^{\alpha} y^{\beta-1} z^{\gamma} = \lambda b$$

$$3. \quad \gamma k x^{\alpha} y^{\beta} z^{\gamma-1} = \lambda c$$

$$4. \quad ax + by + cz = d.$$

Multiplying equation 1 by b and equation 2 by a we find

$$a\alpha b k x^{\alpha-1} y^{\beta} z^{\gamma} = a\beta k x^{\alpha} y^{\beta-1} z^{\gamma}$$

$$y = \frac{a\beta}{b\alpha} x.$$

Now that we have solved for y in terms of x we can use the same procedure to solve for z in terms of x. Multiplying 1 by c and 3 by a reveals

$$c\alpha b k x^{\alpha-1} y^{\beta} z^{\gamma} = a\gamma k x^{\alpha} y^{\beta} z^{\gamma-1}$$

$$z = \frac{a\gamma}{c\alpha}x$$

Now that we have solved for both y and z, we can substitute the results into equation 4 and solve for x in terms of the given constants.

$$ax + b\frac{a\beta}{b\alpha}x + c\frac{a\gamma}{c\alpha}x = d$$

$$ax(\frac{\alpha+\beta+\gamma}{\alpha}) = d$$

Recall that the numerator in the lefthand side of this last equation is equal to 1.

$$\frac{ax}{\alpha} = d \Rightarrow x = \frac{d\alpha}{a}$$

Applying the equations we found relating x to y and x to z we have $y = \frac{d\beta}{b}$, and $z = \frac{d\gamma}{c}$. The function F has a critical point at $\left(\frac{d\alpha}{a}, \frac{d\beta}{b}, \frac{d\gamma}{c}, \lambda\right)$; therefore, f reaches its maximum with respect to the constraint function at $\left(\frac{d\alpha}{a}, \frac{d\beta}{b}, \frac{d\gamma}{c}\right)$.