MATH 223

Some Notes on Assignment 15

Exercises 7, 8ab, 10, 11, and 12 in Chapter 5.

7: *Solution:* a)

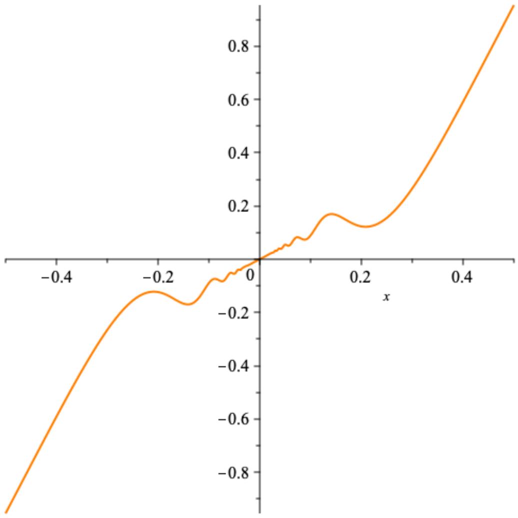


Figure 1: The function $f(x)$ plotted over the interval $[-.5, .5]$.

b) In the case that x is 0, $f(x) = 0$ and both proposed inequalities are true. In the case that x does not equal 0, recall that the $\sin x$ function is bounded by 1 and -1 for all values of x. We then have

$$-2x^2 \leq 2x^2 \sin\frac{1}{x} \leq 2x^2.$$

Adding x to each term in the inequality gives

$$x - 2x^2 \leq x + 2x^2 \sin\frac{1}{x} \leq x + 2x^2.$$

$$x - 2x^2 \leq f(x) \leq x + 2x^2.$$

The value $2x^2$ is nonnegative for all x, so

$$-2x^2 \leq 0 \leq 2x^2.$$

Adding an x to each term in the previous inequality gives

$$x - 2x^2 \leq x \leq x + 2x^2.$$

c) Consider the value of $f'(x)$ at $x_1 = \frac{1}{2k\pi}$ and $x_2 = \frac{1}{(2k+1)\pi}$ where k is a nonzero integer. We have

$$f'(x_1) = 1 + \frac{4}{2k\pi}\sin 2k\pi - 2\cos 2k\pi.$$

Note that wherever x is an even multiple of π, $\sin x$ is 0 and $\cos x$ is 1.

$$f'(x_1) = -1.$$

At the point x_2 we have

$$f'(x_2) = 1 + \frac{4}{(2k+1)\pi}\sin{(2k+1)\pi} - 2\cos{(2k+1)\pi)}.$$

The $\sin x$ term here will also always be 0, but the $\cos x$ term will be -1.

$$f'(x_2) = 3.$$

Because 0 is not included in the interval $[x_1, x_2]$, $f'(x)$ is always continuous on this interval. The Intermediate Value Theorem promises that there exists a point c included in $[x_1, x_2]$ such that $f'(c) = 0$. For any arbitrary neighbourhood of 0, $\mathcal{M}$, there are infinitely many values of k for which x_1, x_2, and c are all included in $\mathcal{M}$; therefore, any $\mathcal{M}$ includes infinitely many x such that $f'(x) = 0$.

d) Wherever x is non zero we, $f(x)$ is the composition of differentiable functions and is therefore differentiable. At $x = 0$ we have

$$f'(0) = \lim_{h\to 0} m \frac{h + 2h^2 \sin \frac{1}{h}}{h}$$

$$f'(0) = \lim_{h\to 0} 1 + 2h \sin \frac{1}{h}$$

The sin function is bounded above and below by 1 and -1 so we can write

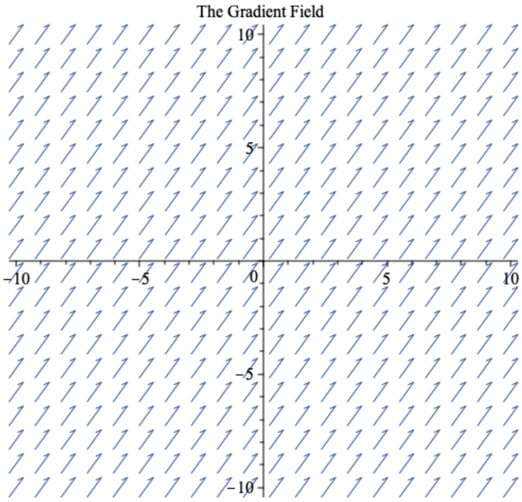
$$\lim_{h\to 0} 1 - 2h \leq \lim_{h\to 0} 1 + 2h \sin \frac{1}{h} \leq \lim_{h\to 0} 1 + 2h$$

$$1 \leq f'(0) \leq 1 \Rightarrow f'(0) = 1.$$

We could use the method applied in part c to show that f' achieves the value 1 at infinitely many places in an arbitrary neighbourhood of 0. The function f' then achieves the values 0 and 1 in any arbitrary neighbourhood of 0 and cannot be continuous at $x = 0$. If f' is not continuous at 0 it is also not differentiable.

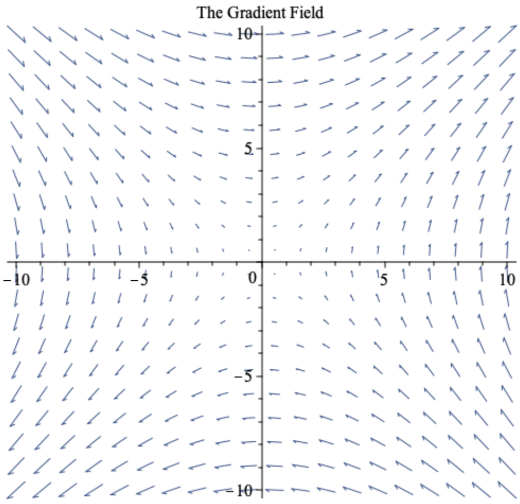
Solution: **8ab:** a) $\nabla f = (2, 3)$

The Gradient Field



The field of gradient vectors for the function $f(x, y) = 2x + 3y$.

b) $\nabla f = (2x, 2y)$



The field of gradient vectors for the function $f(x, y) = 2\,x\,y$.

10: *Solution:* atBarrier 10. a)

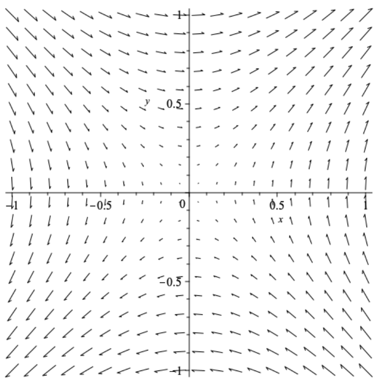


Figure 2: The vector field $\mathbf{F}(x, y) = (y, x)$.

b)

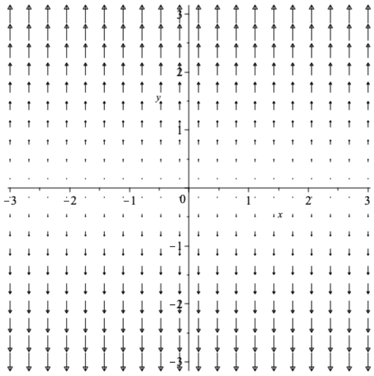


Figure 3: The vector field $\mathbf{F}(x, y) = (0, y)$.

c)

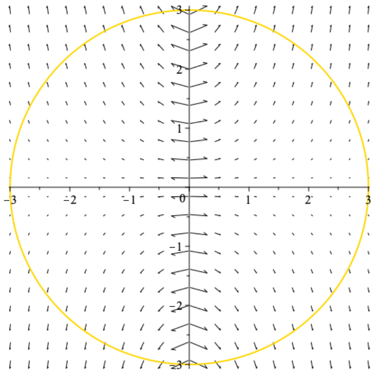


Figure 4: The vector field $\mathbf{F}(x, y) = \nabla(\ln x^2 + y^2)$.

11: *Solution:* From Example 3 in the Gradient Field section of the text, we know that if a vector field $\mathbf{F}(x, y) = (g(x, y), h(x, y))$ does not have the property $g_y = h_x$, then it is not a Gradient Field. Thus, in order to prove that a given Vector field is not a gradient we must show that the partial derivatives are unequal.
a) If $\mathbf{F}(x, y) = (2x^2y^2, x^3e^y)$ then the partial derivatives of the component functions are $(2x^2y^2)_y = 4x^2y$ and $(x^3e^y)_x = 3x^2e^y$. The partial derivatives with respect to x and y are not equal so this is not a gradient field.
b) If $\mathbf{F}(x, y) = (\sin y, \sin x)$ then the partial derivatives of the component functions are $(\sin y)_y = \cos y$ and $(\sin x)_x = -\cos x$. The partial derivatives are not equivalent so The given vector field is not a gradient field.
c) If $\mathbf{F}(x, y) = (xe^{x^2y^2}, ye^{x^2y^2})$, then the partial derivatives of interest are $(xe^{x^2y^2})_y = 2x^3ye^{x^2y^2}$ and $(ye^{x^2y^2})_x = 2xy^3e^{x^2y^2}$. The partial derivatives with respect to x and y are not equivalent, so $\mathbf{F}$ is not a gradient field.

12: *Solution:* For $\mathbf{F}(x, y, z)$ to be a gradient field, each second order mixed partial derivative must be equal regardless of order of differentiation. Not only does G_y need to be equal to H_x, but we must have $G_z = K_x$ and $H_z = K_y$. Differentiating G with respect to y and K with respect to x we find $G_z = y$ and $K_x = 2x$, so $G_z \neq K_x$.