MATH 223

Some Notes on Assignment 12

Exercises 26, 27, 28, and 29 in Chapter 4. (Old Exercises 19, 20, 21, 22)

26. Find the directional derivative of $f(x, y) = x^3\sqrt{y}$ at (7,25) in the direction toward (3,-4).

Solution: If $f(x, y) = x^3\sqrt{y}$, then the gradient, $\nabla f(x, y) = [3x^2\sqrt{y}, \frac{x^3}{2\sqrt{y}}]$ is the 1 by 2 vector containing the partial derivatives of f with respect x and y. To find the derivative of f in the direction of of $\mathbf{v} = (3, 4)$ we need only find the product $\nabla f(7, 25) \cdot \mathbf{u}$ where $\mathbf{u}$ is the unit vector with the same direction as $\mathbf{v}$.

$$\mathbf{v} = (3, -4) \rightarrow \mathbf{u} = (3, -4)(\frac{1}{\sqrt{3^2 + 4^2}}) = (\frac{3}{5}, \frac{-4}{5})$$

$$f_{\mathbf{u}}(7, 25) = \nabla f(7, 25)\mathbf{u} = [735, \frac{343}{10}](\frac{3}{5}, \frac{-4}{5}) = \frac{10339}{25} = 413.56$$

27: Find the directional derivative of $f(x, y, z) = xy^2z^3$ at (1,6,2) in the direction toward (3, -1, -1).

Solution: To find a derivative of $f(x, y, z)$ in the direction of $\mathbf{v} = (3, -1, -1)$, we must find the gradient of f at $(1, 6, 2)$ and a unit vector in the direction of $\mathbf{v}$. If we take the partial derivatives with respect to each variable of f we find the gradient to be $\nabla f(x, y, z) = [y^2z^3, 2xyz^3, 3xy^2z^2]$. Evaluated at the point $(1, 6, 2)$ the gradient is $[288, 96, 432]$. The unit vector in the direction of v is $\frac{\mathbf{v}}{|\mathbf{v}|} = (\frac{3}{11}, \frac{-1}{11}, \frac{-1}{11})$. From Theorem 4.3.1. we have

$$f_u(1, 6, 2) = [288, 96, 432](\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}) = \frac{336}{\sqrt{11}} \approx 101.308$$

28: At the point (3,2), a certain function f has a directional derivative of $\frac{288}{5}$ in the direction (-3,4) and a directional derivative of $-\frac{36}{13}$ in the direction (-12,5). Find the gradient of f at (3,2).

Solution: Let $\mathbf{v}_1 = (-3, 4)$, and $\mathbf{v}_2 = (-12, 5)$. If the directional derivative of $f(x, y)$ at $(3, 2)$ is $\frac{288}{5}$ in the direction of $\mathbf{v}_1$, and $\frac{-36}{13}$ in the direction of v_2 then we have

$$\nabla f(3, 2) \cdot \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{288}{5},$$

$$\nabla f(3, 2) \cdot \frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \frac{-36}{13}.$$

If we calculate the value of the unit vectors $\frac{\mathbf{v}}{|\mathbf{v}|}$, and expand the gradient of $f(3, 2)$ to be the 1 by 2 matrix of partial derivatives we get

$$[f_x(3, 2), f_y(3, 2)](\frac{-3}{5}, \frac{4}{5}) = \frac{288}{5},$$

$$[f_x(3, 2), f_y(3, 2)](\frac{-12}{13}, \frac{5}{13}) = \frac{-36}{13},$$

$$-3f_x(3,2) + 4f_y(3,2) = 288,$$
$$-12f_x(3,2) + 5f_y(3,2) = -36.$$

These last two equations can be rewritten as a system of linear equations of the form

$$\begin{pmatrix} -3 & 4 \\ -12 & 4 \end{pmatrix} \begin{pmatrix} f_x(3,2) \\ f_y(3,2) \end{pmatrix} = \begin{pmatrix} 288 \\ -36 \end{pmatrix}$$

Using either a linear algebra software or row reduction we find $f_x(3,2) = 48$ and $f_y(3,2) = 108$. The gradient of $f(3,2)$ is then $\nabla f(3,2) = (48, 108)$.

29 The real-valued function f is differentiable at a certain point $\mathbf{x}$ in $\mathbb{R}^3$ with the following known directional derivatives: $-1/\sqrt{14}$ in the direction (-2,3,1), 0 in direction (-5,-1,8) and $2\sqrt{3}$ in direction (1,1,1). Find the directional derivative in the direction (3, 4, -5).

Solution: If the directional derivative of $f(x,y,z)$ at a particular point is $\frac{-1}{\sqrt{14}}$ in the direction $(-2,3,1)$, 0 in the direction $(-5,-1,8)$, and $2\sqrt{3}$ in the direction $(1,1,1)$, by Theorem 4.3.1 and the definition of directional derivatives we have

$$\nabla f(\mathbf{x}) \cdot \frac{(-2,3,1)}{|(-2,3,1)|} = \frac{-1}{\sqrt{14}},$$

$$\nabla f(\mathbf{x}) \cdot \frac{(-5,1,8)}{|(-5,1,8)|} = 0,$$

$$\nabla f(\mathbf{x}) \cdot \frac{(1,1,1)}{|(1,1,1)|} = 2\sqrt{3}.$$

Notice that each of these equations can be simplified by multiplying by the magnitude of the respective direction vectors to get

$$\nabla f(\mathbf{x}) \cdot (-2,3,1) = -1,$$

$$\nabla f(\mathbf{x}) \cdot (-5,1,8) = 0,$$

$$\nabla f(\mathbf{x}) \cdot (1,1,1) = (2)(\sqrt{3})(\sqrt{3}) = 6$$

If the gradient of $f(\mathbf{x})$ is expanded to be the 1 by 3 matrix of partial derivatives at $\mathbf{x}$ the system of equations becomes

$$-2f_x(\mathbf{x}) + 3f_y(\mathbf{x}) + f_z(\mathbf{x}) = -1,$$

$$-5f_x(\mathbf{x}) - f_y(\mathbf{x}) + 8f_z(\mathbf{x}) = 0,$$

$$f_x(\mathbf{x}) + f_y(\mathbf{x}) + f_z(\mathbf{x}) = 6.$$

Rewritten in matrix form this system is

$$\begin{pmatrix} -2 & 3 & 1 \\ -5 & -1 & 8 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_x(\mathbf{x}) \\ f_y(\mathbf{x} \\ f_z(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

Using a linear algebra software or row reduction to solve for each partial derivative reveals $f_x(\mathbf{x}) = 3$, $f_y(\mathbf{x}) = 1$, and $f_z(\mathbf{x}) = 1$. The gradient of f at $\mathbf{x}$ is $\nabla f(\mathbf{x}) = (3,1,2)$. Then the directional derivative in the direction(3,4,-5) is $(3,1,2) \cdot \frac{(3,4,-5)}{\sqrt{3^2+4^2+(-5)^2}} = \frac{3}{5\sqrt{2}}$