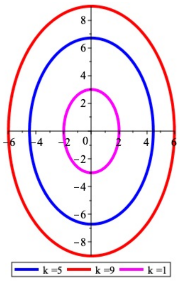
MATH 223

Some Notes on Assignment 6

Chapter 3: 10, 17 abc, 19ac, 20 and 22.

10: Sketch and discuss the level curves $\frac{x^2}{4} + \frac{y^2}{9} = k$ for $k = 9, 5, 1, 0, -1$
Solution: At each level, k, the set of all points for which $f(x, y) = k$ is an ellipse centered at the origin. When $k = 0$ the level set is the origin, and when $k = -1$ the level set is empty.



17abc: Find f_x and f_y or each of the following:
a: $f(x, y) = \sin xy$
Solution: To take the partial derivative with respect to x we can treat y as a constant and the derivative becomes a simple application of the Chain Rule: $f_x(x, y) = y \cos xy, f_y x, y) = x \cos xy$
b: $f(x, y) = \tan e^x$ *Solution:* Recall from single variable calculus that the derivative of $\tan x$ is $\sec^2 x$ and the derivative of e^x is e^x. If we hold y constant and take the derivative with respect to x, the Chain Rule gives $f_x x, y) = \frac{e^x}{(\cos^2 e^x)}$ As there is no y included in $f(x, y)$, the entire expression is a constant if x is treated as a constant. The derivative $f_y x, y)$ is then 0.

c: $f(x, y) = \frac{\arctan y}{x}$ *Solution:* If we treat y as a constant then the term $\arctan y$ is a constant. Then the partial derivative with respect to x is $(\arctan y)(-x^{-2})$. From single variable calculus we know that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$. The partial derivative of $f(x, y)$ with respect to y is then $(\frac{1}{x})\frac{1}{1+y^2}$.

19ac For each of these functions f, determine $f_x(2, 3)$ and $f_y(2, 3)$:

a: $f(x, y) = x^3 + 4xy - y^2$
Solution: If $f(x, y) = x^3 + 4xy - y^2$ then we have

$$f_x(x, y) = 3x^2 + 4y \to f_x(2, 3) = 24 \text{ and}$$

$$f_y(x, y) = 4x - 2y \to f_y(2, 3) = 2$$

c $f(x,y)=\frac{2x-3y}{3x+2y}$ *Solution:* Because $f(x,y)=\frac{2x-3y}{3x+2y}$ contains an x term and y term in both the numerator and the denominator, we will need to apply The Quotient Rule to find both partial derivatives.

$$f_x(x,y)=\frac{13y}{(3x+2y)^2}\to f_x(2,3)=\frac{13}{48}\text{ and}$$
$$f_y(x,y)=\frac{-13x}{(3x+2y)^2}\to f_y(2,3)=\frac{-13}{72}$$

20: Josh's utility function for two particular goods has the form $U(x,y)=(x+3)^2(y+2)^3$. Find the marginal utility functions U_x and U_y and evaluate them if $x=4, y=4$.
Solution: If the utility of goods x and y is $U(x,y)=(x+3)^2(y+2)^3$ then the marginal utility of good x is $U_x(x,y)=(y+2)^3 2(x+3)$ and the marginal utility of good x at $(4,4)$ is 3024. The marginal utility of good y is $u_y(x,y)=(x+3)^2 3(y+2)^2$; evaluated at $(4,4)$ we have $U_y(4,4)=5292$.

22: A thin, homogeneous metal rod lying along the horizontal axis from 0 to L has a nonuniform temperature. Heat (thermal energy) transfers from regions of higher temperature to regions of lower temperature. Under certain conditions the function $u(x,t)$ which gives the temperature at position x and time t obeys the **diffusion equation** $u_{xx}=4u_t$. Show that function

$$u(x,t)=\frac{e^{-x^2/t}}{\sqrt{x}}, t>0$$

satisfies the diffusion equation.

Solution: To see whether or not the function $u(x,t)$ satisfies the diffusion equation we need to find the second order partial derivative u_{xx} and the first order partial derivative u_t. If we factor out $\frac{1}{\sqrt{t}}$ and differentiate with respect to x using the Product Rule we find

$$u_x(x,t)=\frac{-2xe^{\frac{-x^2}{t}}}{t^{\frac{3}{2}}}.$$

Now we can factor out $\frac{1}{t^{\frac{3}{2}}}$ and use the Product Rule to find

$$u_{xx}(x,t)=e^{\frac{-x^2}{t}}(\frac{4x^2}{t^{\frac{5}{2}}}-\frac{2}{t^{\frac{3}{2}}}).$$

Now we need to use the Quotient Rule on $u(x,t)$ to find $u_t(x,t)$.

$$u_t(x,t)=e^{\frac{-x^2}{t}}(\frac{x^2}{t^{\frac{5}{2}}}-\frac{1}{2t^{\frac{3}{2}}})=4u_{xx}(x,t).$$

The equation does satisfy the diffusion equation.