MATH 223
Some Notes on Assignment 4
Chapter 2: Exercises 35, 40, 42, 43 and Problem A

35: For a parametrized curve $\mathbf{x}(t), a \le t \le b$, the **arc length L** is the integral of its speed:

$$\mathbf{L} = \int_a^b |\mathbf{v}(t)|\, dt = \int_a^b |\mathbf{x}'(t)|\, dt$$

(a): If the speed is constant, show that arc length is given by the classic distance = rate × time formula.
Solution: If speed is constant, then $|\mathbf{x}'(t)| = r$ for some constant r in which case $\mathbf{L} = \int_a^b r\, dt = r(b-a) =$ rate × time.

(b): Show that the curve with parametrization $\mathbf{g}(t) = (a\sin\omega t, a\cos\omega t)$ with $a > 0$ traces out, in a clockwise direction a portion of the circle of radius a centered at the origin with speed $a\omega$. We call the constant ω the **angular speed** .
Solution: $|\mathbf{g}(t)| = \sqrt{a^2\sin^2\omega t + a^2\cos^2\omega t)} = \sqrt{a^2} = a$ so $\mathbf{g}(t)$ lies on the circle of radius a centered at the origin for all t. At $t = 0$, we are at the point $(0, a)$; as t increases from 0, $a\sin\omega t$ will increase and $a\cos\omega t$ will decrease so we move in a clockwise direction. We trace out the entire circle as t goes from 0 to 2π.
Now speed $= |\,\mathbf{g}'(t)| = |(a\omega\cos\omega t, -a\omega\sin\omega t| = \sqrt{a^2\omega^2\cos^2\omega t + a^2\omega^2\sin^2\omega t} = \sqrt{a^2\omega^2} = a\omega$.

(c) Describe the difference in motion between the parametrizations $\mathbf{x}(t) = (\sin t, \cos t), 0 \le t \le 2\pi$ and $\mathbf{y}(u) = (\sin u, \cos u), 0 \le u \le 6\pi$.
Solution: Both trace out the unit circle in the same direction and with the same speed but $\mathbf{y}$ traces out the circle 3 times while $\mathbf{x}$ only traces it out once.

40: For the position function $\mathbf{p}(t) = (\cos t, \sin t)$ with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, what is the geometric relationship between $\mathbf{p}$ and $\mathbf{v}$? Between $\mathbf{p}$ and $\mathbf{a}$? Between $\mathbf{v}$ and $\mathbf{a}$? *Solution:* $\mathbf{v} = (-\sin t, \cos t)$ and $\mathbf{a} = (-\cos t, -\sin t) = -\mathbf{p}$ so $\mathbf{p}$ and $\mathbf{v}$ are orthogonal to each other and $\mathbf{a}$ points in the opposite direction of $\mathbf{p}$ so $\mathbf{a}$ and $\mathbf{v}$ are also orthogonal.

42: Find all differentiable vector valued functions $\mathbf{p}$ such that $\mathbf{p}'(t) = (te^t, \cos t, 2t\sqrt{1+t^2})$ and $\mathbf{p}(0) = (0,0,0)$
Solution : If $\mathbf{p}'(t) = (te^t, \cos t, 2t\sqrt{1+t^2})$ then $\mathbf{p}(t)$ have the form $\mathbf{p}(t) = ((t-1)e^t + C_1, \sin t + C_2, \frac{2}{3}(1+t^2)^{\frac{3}{2}} + C_3)$ where C_1, C_2, and C_3 are all real constants. Given that we know $\mathbf{p}(0) = (0,0,0)$ we can solve for the undetermined constants.

$$(0-1)e^t + C_1 = 0 \Rightarrow C_1 = 1$$

$$\sin 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$\frac{2}{3}(1+t^2)^{\frac{3}{2}} + C_3 = 0 \Rightarrow C_3 = -\frac{2}{3}$$

Hence $\mathbf{P}(t) = ((t-1)e^t + 1, \sin t, \frac{2}{3}(1+t^2)^{\frac{2}{3}} - \frac{2}{3}$.

42: If $\mathbf{p}'(t) = (te^t, \cos t, 2t\sqrt{1+t^2})$ then $\mathbf{p}(t)$ have the form $\mathbf{p}(t) = ((t-1)e^t + C_1, \sin t + C_2, \frac{2}{3}(1+t^2)^{\frac{3}{2}} + C_3)$ where C_1, C_2, and C_3 are all real constants. [Use integration by parts on the first component with $u = te^t, dv = dt$, and substitution $u = 1 + t^2$ for the third component.

Given that we know $\mathbf{p}(0) = (0,0,0)$ we can solve for the undetermined constants.

$$(0-1)e^t + C_1 = 0 \Rightarrow c_1 = 1$$

$$\sin 0 + C_2 = 0 \Rightarrow c_2 = 0$$

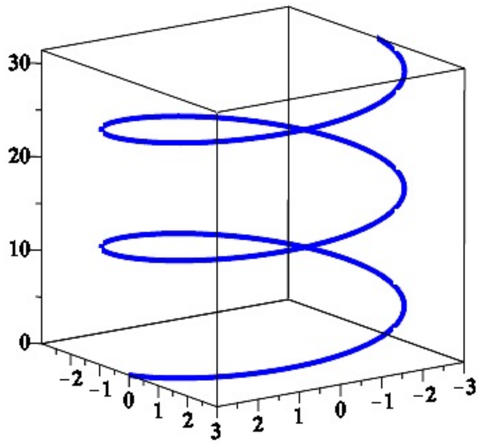
$$\frac{2}{3}(1+t^2)^{\frac{3}{2}} + C_3 = 0 \Rightarrow c_3 = -\frac{2}{3}$$

Hence $\mathbf{P}(t) = \left((t-1)e^t + 1, \sin t, \frac{2}{3}(1+t^2)^{\frac{2}{3}} - \frac{2}{3}\right)$.

43: Determine $\int \mathbf{p}(t)dt$ for each of the following vector-valued functions:

1. $\mathbf{p}(t) = (4t^3, 3t^2, 2t, 1, 0)$. *Solution:* $(t^4, t^3, t, 0) + (C1, C2, C3, C4)$
2. $\mathbf{p}(t) = (t^2, 1 + 7t, 2/t)$. *Solution:* $\left(\frac{t^3}{3}, t + 7\frac{t^2}{2}, 2\ln|t|\right) + (C1, C2, C3)$
3. $\mathbf{p}(t) = (e^t, \tan t, \ln t)$. *Solution:* $(e^t, -\ln\cos t, t\ln t - t) + (C1, C2, C3)$

Problem A: Let $\mathbf{f}(t) = (a\cos t, a\sin t, bt)$ with a and b nonzero constants. Sketch the graph of this curve (a **helix**) for $0 \le t \le 5\pi$. Show that the speed is constant and the velocity vector is always orthogonal to the vector $\mathbf{r}(t) = (a\cos t, a\sin t, 0)$.
Solution: The velocity vector is $\mathbf{v} = (-a\sin t, a\cos t, b)$ so speed $= |\mathbf{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$
and $\mathbf{v}\cdot\mathbf{r} = -a^2\cos t\sin t + a^2\cos t\sin t + 0t = 0$.



The Helix with $a = 3$and $b = 2$.