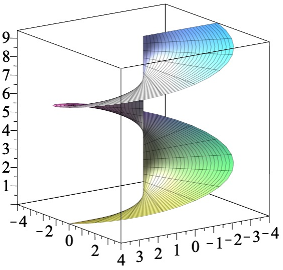
MATH 223: Multivariable Calculus



Class 9: September 30, 2022

Handouts

- Notes on Assignment 8
- Assignment 9

Announcements

Exam 1: Next Monday, 7 PM - No Time Limit

No Books, Notes, Computers, etc.

Focus on Chapters 2 and 3 **No Class Next Wednesday Make Up Class Thursday Evening**

Tangent Planes To Surfaces

(I) $f : \mathcal{R}^2 \to \mathcal{R}^1$**, a a point in** $\mathcal{R}^2$

Tangent plane to graph of f at $(\mathbf{a}, f(\mathbf{a}))$:

$$T(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

(II): $f : \mathcal{R}^2 \to \mathcal{R}^3$

$$\sigma(s,t) = (f(s,t), g(s,t), h(s,t))$$

$\sigma_s(s,t) = (f_s, g_s, h_s)$ and. $\sigma_t(s,t) = (f_t, g_t, h_t)$

Tangent Plane at $\sigma(\mathbf{a})$:

$$\sigma(\mathbf{a}) + (s,t)\begin{pmatrix} f_s(\mathbf{a}) & g_s(\mathbf{a}) & h_s(\mathbf{a}) \\ f_t(\mathbf{a}) & g_t(\mathbf{a}) & h_t(\mathbf{a}) \end{pmatrix}$$

Note: $1 \times 3 + (1 \times 2)(2 \times 3)$

Writing vectors vertically: $\sigma = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \sigma' = \begin{pmatrix} f' \\ g' \\ h' \end{pmatrix}$

Tangent Plane: $T\begin{pmatrix} s \\ t \end{pmatrix} = \sigma(\mathbf{a}) + \sigma'(\mathbf{a})\begin{pmatrix} s \\ t \end{pmatrix}$

Example: $f(x,y,z) = \frac{x^2 y}{z}$
Note: $f : \mathcal{R}^3 \to \mathcal{R}^1$ so GRAPH lives in $\mathcal{R}^4$.
Find Equation of Tangent Hyperplane at $\mathbf{a} = (-3,4,2)$

$$f_x(x,y,z) = \frac{2xy}{z}$$

$$f_y(x,y,z) = \frac{x^2}{z} \;\; so \nabla f(x,y,z) = \left(\frac{2xy}{z}, \frac{x^2}{z}, -\frac{x^y}{z^2}\right)$$

$$f_z(x,y,z) = -\frac{x^y}{z^2}$$

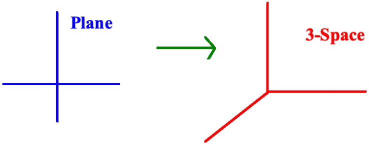
at $\mathbf{a} = (-3,4,2)$: $f(\mathbf{a}) = \frac{(-3)^2 \times 4}{2} = 18$

$$\nabla f(\mathbf{a}) = \left(\frac{(2)(-3)(4)}{2}, \frac{(-3)^2}{2}, \frac{-(-3)^2(4)}{2}\right) = \left(-12, \frac{9}{2}, -9\right)$$

Equation of Tangent Hyperplane is

$$w = 18 + \left(-12, \frac{9}{2}, -9\right) \cdot (x+3, y-4, z-2)$$

Parametrized Surfaces



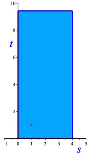
Function from $\mathcal{R}^2 \to \mathcal{R}^3$

Domain	Patch in Plane
Image	Surface in Space
Graph	Lives in $\mathcal{R}^5$

Need for Parametrizations: Graph of $f : \mathcal{R}^1 \to \mathcal{R}^1$ is a curve but not every curve is the graph of such a function

Similarly, graph of $f : \mathcal{R}^2 \to \mathcal{R}^1$ is a surface but not every surface is the graph of such a function.

Example: $\sigma(s,t) = (s\cos t, s\sin t, t), 0 \le s \le 4, 0 \le t \le 3\pi$

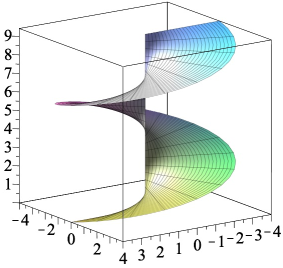


$$\text{Point: } (1, \pi/4) \text{ so } \sigma(1,\pi/4) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$$

$$\sigma_s(s,t) = (\cos t, \sin t, 0) \text{ and } \sigma_t(s,t) = (-s\sin t, s\cos t, 1)$$

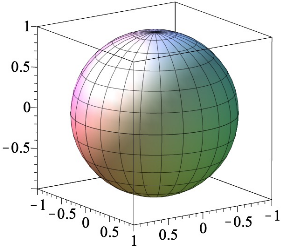
At $\left(1, \frac{\pi}{4}\right)$, representation of the tangent plane is

$$\sigma\left(1, \frac{\pi}{4}\right) + \sigma_s\left(1, \frac{\pi}{4}\right) s + \sigma_t\left(1, \frac{\pi}{4}\right) t$$



Parametrize Unit Sphere

$$\sigma(s,t) = (\cos t \cos s, \sin t \cos s, \sin s), 0 \le s \le 2\pi, 0 \le t \le 2\pi$$



$$x = \cos t \cos s, y = \sin t \cos s, z = \sin s$$

$$\begin{aligned} x^2 + y^2 + z^2 &= \cos^2 t \cos^2 s + \sin^2 t \cos^2 s + \sin^2 s \\ &= \cos^2 s(\cos^2 t + \sin^2 t) + \sin^2 s \\ &= \cos^2 s + \sin^2 s = 1 \end{aligned}$$

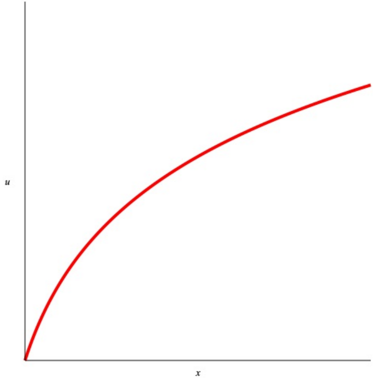
Utility

Utility = happiness, satisfaction, pleasure, usefulness

$u(x), x \geq 0$

Typical Assumptions: u is increasing, concave down function

("decreasing returns to scale")



Example: $u(x) = x^{1/3}$ so $u'(x) = \frac{1}{3x^{2/3}}, u''(x) = -\frac{2}{9}x^{-5/3}$

Example: 2 Goods with $u(x, y) = \sqrt[3]{xy}$
Each unit of x costs \$35 and each unit of y costs \$80
We have \$D to spend: Budget Constraint: $35x + 80y = D$
Goal: Maximize Utility:

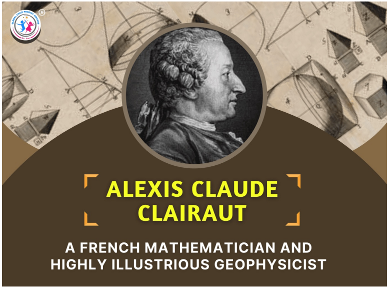
$$80y = D - 35x \text{ so } y = \frac{D - 35x}{80}$$

$$u(x, y) = f(x) = \sqrt[3]{\frac{x(D - 35x)}{80}}$$

f is maximized when $\frac{x(D-35x)}{80}$ is maximized.
$G(x) = x(D - 35x). = Dx - 35x^2.$ has $G'(x) = D - 70x$ and
$G''(xx) = -70$ Hence there is a maximum when $x = D/70$
Then $y = \frac{D-35(D/70)}{80} = D/160$

Clairaut's Theorem on Equality of Mixed Partials

If f_{xy} and f_{yx} are continuous at $\mathbf{a}$, then $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$



May 7, 1713 – May 17, 1765

Clairaut's Theorem on Equality of Mixed Partials

If f_{xy} and f_{yx} are continuous at $\mathbf{a}$, then $f_x y(\mathbf{a}) = f_y x(\mathbf{a}$

$$f(x,y) = \begin{Bmatrix} 2xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{Bmatrix}$$

It Turns. Out That

$$f_{xy}(0,0) = -2$$
$$f_{yx}(0,0) = +2$$

Mixed Partials Are Not Equal